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Cambridge Philosophical Society.

VOLUME VI.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 25, 1886.

ANNUAL GENERAL MEETING.

PROFESSOR FOSTER, PRESIDENT, IN THE CHAIR.

THE following were elected officers and new members of the Council for the ensuing year:—

President:

Mr Trotter.

Vice-Presidents:

Prof. Babington, Prof. Adams, Prof. Foster.

Treasurer:

Mr J. W. Clark.

Secretaries:

Mr Glazebrook, Mr Vines, Mr Larmor.

New Members of Council:

Prof. Liveing, Mr Forsyth, Mr Marr, Mr Pattison-Muir.

The following communications were made to the Society.

(1) *On the Potentials of the surfaces formed by the revolution of Limaçons and Cardioids about their axes.* By A. B. Basset, M.A.

1. The first part of this paper deals with the potentials of the surfaces formed by the revolution about their axes of the curves which are inverses of ellipses with respect to a focus, and which for shortness I shall call elliptic limaçons, to distinguish them from the orthogonal system of curves which are the inverses with respect to the same focus of the system of confocal hyperbolas, and which may be called hyperbolic limaçons.

The potential of any distribution of electricity upon the surface of a prolate spheroid can be expressed, as is well known, by means of a series of spheroidal harmonics; hence the potential of the surface formed by the revolution about its axis of an elliptic limaçon can be expressed in a similar manner by means of the method of inversion. The same result can also be obtained independently by employing a transformation of Laplace's Equation due to C. Neumann*, which is reproduced in the 19th volume of the *Quarterly Journal*, pages 349 and 350.

The potential of a paraboloid of revolution, and thence by inversion that of the surface formed by the revolution of a cardioid about its axis, which is dealt with in the second part, can be obtained either by regarding a parabola as a limiting form of an ellipse, or independently; and it will be found that the result assumes the form of a definite integral involving Bessel's Functions.

The fact that the potentials of paraboloids and cardioids of revolution can be expressed by means of Bessel's Functions, appears to have been noticed by F. G. Mehler of Danzig, and Carl Baer (see Heine, *Handbuch der Kugelfunctionen*, Vol. II. pp. 174 and 292), but I have not as yet been able to obtain any of their papers.

Part I.

2. Choosing the axis of x as the axis of revolution, let x, ρ, θ be cylindrical co-ordinates of a point, and let ξ and η be conjugate functions of x and ρ such that

$$x - \iota\rho = 2c\cos^2\tfrac{1}{2}(\xi + \iota\eta),$$

then it is known that the curves $\eta = \text{const.}$, $\xi = \text{const.}$ are a family of confocal prolate spheroids, and hyperboloids of two sheets; the

* *Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe.* Halle, 1864.

origin being that focus which lies on the negative or left-hand side of the centre. Also, if we put

$$\nu = \cosh \eta, \quad \mu = \cos \xi,$$

it is known that the potential V, of any distribution of electricity upon a spheroid, can be expressed by means of the following series, viz.:

$$V = \Sigma\Sigma A_n \frac{Q_n^m(\nu)}{Q_n^m(\gamma)} P_n^m(\mu) \sin(m\theta + \alpha_m)$$

at an external point; and

$$V = \Sigma\Sigma A_n \frac{P_n^m(\nu)}{P_n^m(\gamma)} P_n^m(\mu) \sin(m\theta + \alpha_m)$$

at an internal point; where γ is the value of ν at the surface, and the functions P_n^m and Q_n^m are what Todhunter, translating Heine, calls associated functions of the first and second kinds respectively.

The functions of the second kind can be expressed either in a series of powers of $1/\nu$, or by means of definite integrals; see Heine, *Handbuch der Kugelfunctionen*, Vol. I. ch. iv.; *Messenger of Mathematics*, Vol. XIII. p. 147.

If we invert with respect to the origin, the spheroid will invert into the surface formed by the revolution about its axis, of an elliptic limaçon; whence it follows that the potential at all points outside this latter surface will be of the form

$$V = \frac{c}{r} \Sigma\Sigma A_n \frac{P_n^m(\nu)}{P_n^m(\gamma)} P_n^m(\mu) \sin(m\theta + \alpha_m) \quad \ldots\ldots(1),$$

and at an internal point

$$V = \frac{c}{r} \Sigma\Sigma A_n \frac{Q_n^m(\nu)}{Q_n^m(\gamma)} P_n^m(\mu) \sin(m\theta + \alpha_m) \quad \ldots\ldots(2),$$

where μ and ν are respectively equal to $\cos\xi$ and $\cosh\eta$ as before; but

$$x + \iota\rho = 2c \sec^2 \tfrac{1}{2}(\xi + \iota\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $2c$ is the constant of inversion.

3. The foregoing results can also be obtained without recourse to the method of inversion, by means of Neumann's transformation of Laplace's equation.

It therefore follows, that the solution of any problem, which consists in finding the value of a potential function, which has a given value at all points of the surface under consideration, is reduced by expanding a given function of μ and θ, in a series of

the form $\Sigma BP_n^m(\mu)\sin(m\theta+\beta)$, where m must be always less than n.

4. Putting $$\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{d\rho}\right)^2=J^2$$

we easily obtain by means of (3),

$$r=\frac{4c}{\nu+\mu},$$

$$J^2=\frac{4c}{r^3(\nu-\mu)},$$

$$J=\frac{4cp}{r^3(\nu^2-1)^{\frac{1}{2}}},$$

where p is the length of the perpendicular from the origin on to the tangent plane to the surface. Also

$$rx=\frac{16c^2(1+\mu\nu)}{(\mu+\nu)^3},$$

$$r\rho=\frac{16c^2\sqrt{1-\mu^2}\sqrt{\nu^2-1}}{(\mu+\nu)^3}.$$

The expansion of r in terms of harmonics is given in Ferrers' *Spherical Harmonics*, Ch. V., viz.

$$r=4c\Sigma_0^\infty(-)^n(2n+1)Q_n(\nu)P_n(\mu)\;\ldots\ldots\ldots\ldots(4).$$

To obtain the expansion of r and $r\rho$, we have

$$\frac{rx}{16c^2}=\frac{1-\nu^2}{(\mu+\nu)^3}+\frac{\nu}{(\mu+\nu)^2}$$

$$=\Sigma_0^\infty(-)^n(2n+1)P_n(\mu)\left\{\tfrac{1}{2}(1-\nu^2)\frac{d^2Q_n}{d\nu^2}-\nu\frac{dQ_n}{d\nu}\right\}$$

$$=-\tfrac{1}{2}\Sigma_1^\infty(-)^n(2n+1)n(n+1)Q_n(\nu)P_n(\mu)\;\ldots\ldots\ldots\ldots(5).$$

Also

$$\frac{r\rho}{16c^2}=\frac{\sqrt{1-\mu^2}\sqrt{\nu^2-1}}{(\mu+\nu)^3}$$

$$=\tfrac{1}{2}\sqrt{1-\mu^2}\sqrt{\nu^2-1}\,\Sigma_1^\infty(-)^n(2n+1)\frac{dQ_n}{d\nu}\frac{dP_n}{d\mu}$$

$$=\tfrac{1}{2}\Sigma_1^\infty(-)^n(2n+1)Q_n^1(\nu)P_n^1(\mu)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

5. We can now obtain the potential of the induced charge, when the surface is placed in a field of electric force, whose

potential is $V_0 + Ax + B\rho \sin\theta$; for denoting the portions of the resulting potential due to each of these three terms by V, V_x, V_ρ, we obtain

$$V = -\frac{4cV_0}{r}\Sigma_0^\infty (-)^n (2n+1)\frac{Q_n(\gamma)}{P_n(\gamma)} P_n(\nu) P_n(\mu) \ldots\ldots\ldots(7),$$

$$V_x = \frac{8Ac^2}{r}\Sigma_1^\infty (-)^n (2n+1)\, n\,(n+1)\frac{Q_n(\gamma)}{P_n(\gamma)} P_n(\nu) P_n(\mu) \ldots(8),$$

$$V_\gamma = -\frac{8Bc^2 \sin\theta}{r}\Sigma_1^\infty (-)^n (2n+1)\frac{Q_n^1(\gamma)}{P_n^1(\gamma)} P_n^1(\nu) P_n^1(\mu) \quad \ldots(9).$$

6. The attraction of the solid can most readily be found from the fact, that the component parallel to any axis (say x) is the potential of a surface distribution of matter of density σl, where l is the x-direction cosine of the normal, and σ the volume-density of the solid which is supposed to be uniform throughout its mass. Hence if X, X' be the values of the x-components at an external and an internal point respectively, the surface conditions are

$$X - X' = 0,$$

$$\frac{dX'}{dn} - \frac{dX}{dn} = 4\pi\sigma l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where dn is an element of the normal drawn outwards.

Now
$$dn = -\frac{d\nu}{J\sinh\eta}$$

$$= -\frac{r^3\, d\nu}{4cp};$$

$$\therefore \quad \frac{dX'}{dn} - \frac{dX}{dn} = \frac{4cp}{r^3}\left(\frac{dX}{d\nu} - \frac{dX'}{d\nu}\right).$$

Let
$$X = \frac{c}{r}\Sigma A_n Q_n(\gamma) P_n(\nu) P_n(\mu),$$

$$X' = \frac{c}{r}\Sigma A_n P_n(\gamma) Q_n(\nu) P_n(\mu).$$

Then if the accents denote differentiation (10) becomes

$$\frac{dX}{d\nu} - \frac{dX'}{d\nu} = \frac{c}{r}\Sigma A_n \{Q_n(\gamma) P_n'(\gamma) - P_n(\gamma) Q_n'(\gamma)\} P_n(\mu)$$

$$= \frac{c}{r(\gamma^2-1)}\Sigma A_n P_n(\mu),$$

since $$Q_n P_n' - Q_n' P_n = 1/(\gamma^2 - 1)^* ;$$

$$\therefore \quad \Sigma A_n P_n(\mu) = \frac{\pi\sigma(\gamma^2-1)r^4 l}{pc^2}.$$

If we put $\quad a = c\,\text{sech}^2\tfrac{1}{2}\eta, \quad b = c\,\text{cosech}^2\tfrac{1}{2}\eta,$

the equation of the meridian curve may be written

$$r - a - b - (a-b)\frac{x}{r} = 0,$$

from which we easily obtain

$$l = \frac{p}{r^2}\left\{x\left[1 + \frac{(a-b)x}{r^2}\right] - a + b\right\},$$

$$m = \frac{p\rho}{r^2}\left\{1 + \frac{(a-b)x}{r^2}\right\}.$$

Therefore

$$l = \frac{64c^3 p}{r^4}\,\frac{2 + \mu\nu - \mu^2}{(\mu+\nu)^4},$$

$$m = \frac{64c^3 p}{r^4}\,\frac{(1-\mu^2)^{\frac{1}{2}}(\nu^2 - 2 - \mu\nu)}{(\nu^2-1)^{\frac{1}{2}}(\mu+\nu)^4};$$

$$\therefore \quad \Sigma A_n P_n(\mu) = 64\pi\sigma c(\gamma^2-1)\,.\,\frac{2+\mu\gamma-\mu^2}{(\mu+\gamma)^4}.$$

Let $$u = \frac{1+\mu\gamma}{(\mu+\gamma)^3};$$

$$\therefore \quad \frac{du}{d\gamma} = \frac{\mu^2 - 2\mu\gamma - 3}{(\mu+\gamma)^4};$$

$$\therefore \quad \frac{2+\mu\gamma-\mu^2}{(\mu+\gamma)^4} = \frac{\gamma}{3(\mu+\gamma)^3} - \frac{1}{3(\mu+\gamma)^2} - \frac{2}{3}\frac{du}{d\gamma}$$

$$= \tfrac{1}{3}\Sigma_0^\infty\,(-)^n(2n+1)\{\tfrac{1}{2}\gamma Q_n''(\gamma) + (n^2+n-1)Q_n'(\gamma)\}\,P_n(\mu),$$

where the accents denote differentiation;

$$\therefore \quad A_n = \tfrac{64}{3}\pi\sigma c(\gamma^2-1)(-)^n(2n+1)\left\{\frac{\gamma}{2}Q_n''(\gamma) + (n^2+n-1)Q'(\gamma)\right\}.$$

7. Similarly if R, R' be the component attractions perpendicular to the axis, we must assume

$$R = \frac{c}{r}\Sigma_1^\infty\,B_n Q_n^1(\gamma)P_n^1(\nu)P_n^1(\mu),$$

$$R' = \frac{c}{r}\Sigma_1^\infty\,B_n P_n^1(\gamma)Q_n^1(\nu)P_n^1(\mu).$$

* *Quarterly Journal*, Vol. xix. p. 352.

Therefore at the surface,

$$\frac{dR}{d\nu}-\frac{dR'}{d\nu}=\frac{c}{r}\Sigma B_n\{Q_n^1(\gamma)P_n'^1(\gamma)-P_n^1(\gamma)Q_n'^1(\gamma)\}P_n^1(\mu)$$

$$=-\frac{c}{r(\gamma^2-1)}\Sigma B_n n(n+1)P_n^1(\mu)^*;$$

$$\therefore\ \Sigma B_n n(n+1)P_n^1(\mu)=-\frac{\pi\sigma(\gamma^2-1)r^4m}{pc^2}$$

$$=-64\pi\sigma c(\gamma^2-1)^{\frac{1}{2}}.\frac{\sqrt{1-\mu^2}(\gamma^2-2-\mu\gamma)}{(\mu+\gamma)^4}.$$

Let $$v=\frac{\sqrt{\gamma^2-1}}{(\mu+\gamma)^3},$$

$$\therefore\ \frac{dv}{d\gamma}=\frac{\mu\gamma-2\gamma^2+3}{\sqrt{\gamma^2-1}\,(\mu+\gamma)^4}.$$

Hence

$$\frac{(\gamma^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}(\gamma^2-2-\mu\gamma)}{(\mu+\gamma)^4}$$

$$=-\frac{(\gamma^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}}{3}\left\{\frac{\gamma}{(\mu+\gamma)^3}+2(\gamma^2-1)^{\frac{1}{2}}\frac{dv}{d\gamma}\right\}$$

$$=-\frac{(\gamma^2-1)^{\frac{1}{2}}}{3}\Sigma_1^\infty(-)^n(2n+1)\left\{\frac{\gamma}{2}Q''+2(\gamma^2-1)^{\frac{1}{2}}Q_n'^1\right\}P_n^1(\mu);$$

$$\therefore\ B_n=\tfrac{64}{3}\pi\sigma c(\gamma^2-1)^{\frac{1}{2}}\Sigma_1^\infty(-)^n\frac{(2n+1)}{n(n+1)}$$

$$\left\{\frac{\gamma}{2}Q_n''(\gamma)+n(n+1)Q_n(\gamma)\right\}.$$

8. When the solid is moving parallel to its axis with velocity V in an infinite liquid, the current function ψ can be determined from the fact, that it is equal to $U\rho\operatorname{cosec}\theta$ when U is the potential of the induced charge, when the solid is placed in a field of electric force whose potential is $-\frac{1}{2}V\rho\sin\theta$.

To prove this, we observe that ψ satisfies the equation

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{d\rho^2}-\frac{1}{\rho}\frac{d\psi}{d\rho}=0 \quad \ldots\ldots\ldots\ldots\ldots(10, a).$$

* See *Quarterly Journal*, Vol. XIX. p. 361, where T_n and t_n are written respectively for P_n^1 and Q_n^1.

Let $\psi = \chi\rho$, then

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{d\rho^2} + \frac{1}{\rho}\frac{d\chi}{d\rho} - \frac{\chi}{\rho^2} = 0 \quad \text{............(10, b)},$$

from which it follows that $\chi \sin\theta$ is a potential function*. Now at the surface of the solid

$$\psi = \tfrac{1}{2} V\rho^2,$$

$$\therefore \chi \sin\theta = \tfrac{1}{2} V\rho \sin\theta.$$

Hence $\chi \sin\theta$ is the potential of the induced charge, when the solid is placed in a field of force whose potential is

$$-\tfrac{1}{2} V\rho \sin\theta,$$

and
$$\psi = \chi\rho = U\rho \operatorname{cosec}\theta.$$

Hence from (9)

$$\psi = \frac{8Vc^2}{r} \Sigma_1^\infty (-)^n (2n+1) \frac{Q_n^{\,1}(\gamma)}{P_n^{\,1}(\gamma)} P_n^{\,1}(\nu) P_n^{\,1}(\mu).$$

PART II.

9. Since a parabola is a limiting form of an ellipse, and consequently a cardioid is a limiting form of an elliptic limaçon, it follows that the results of the preceding portion of this paper may be modified, so as to give the corresponding results in the case of paraboloids and cardioids of revolution.

In order to see how the transition takes place, from a prolate spheroid to a paraboloid, let us consider the case of a symmetrical potential.

The function $Q_n(\nu)$ satisfies the equation,

$$\frac{d}{d\nu}(1-\nu^2)\frac{dQ_n}{d\nu} + n(n+1) Q_n = 0 \quad \text{..............(11)}.$$

* The complete integral of (10, a) may be expressed in the form

$$\psi/\rho = \int_0^\pi \cos\epsilon F(z + \iota\rho\cos\epsilon)\, d\epsilon + \int_0^\infty \cosh\epsilon f(z + \iota\rho\cosh\epsilon)\, d\epsilon.$$

For transforming to polar co-ordinates, $(r, \cos^{-1}\mu)$, it is easily seen that a solution of (10, b) is

$$\chi = \Sigma \left\{ A_n r^n P_n^1(\mu) + \frac{B_n}{r^{n+1}} Q_n^1(\mu) \right\}.$$

Now
$$P_n^1 = \frac{(n+1)}{\pi} \int_0^\pi \{\mu + \sqrt{\mu^2 - 1}\cos\epsilon\}^n \cos\epsilon\, d\epsilon,$$

and
$$Q_n^1 = -n\iota \int_0^\infty \frac{\cosh\epsilon\, d\epsilon}{(\mu + \sqrt{\mu^2-1}\cosh\epsilon)^{n+1}},$$

whence the result at once follows.

Let
$$c(\nu-1)=\eta^2,$$
$$2n(n+1)=\lambda^2 c,$$

nd let c and n increase indefinitely, whilst ν approaches in-lefinitely near to unity, but so that both the quantities η and λ emain finite.

Equation (11) becomes on transformation,

$$\left(1+\frac{\eta^2}{2c}\right)\left(\frac{d^2Q}{d\eta^2}+\frac{1}{\eta}\frac{dQ}{d\eta}\right)-\frac{2n(n+1)}{c}Q=0,$$

whence proceeding to the limit we obtain

$$\frac{d^2Q}{d\eta^2}+\frac{1}{\eta}\frac{dQ}{d\eta}-\lambda^2 Q=0,$$

which is a form of Bessel's equation.

Again, $$Q_n=\int_0^\infty \frac{d\phi}{(\nu+\sqrt{\nu^2-1}\cosh\phi)^{n+1}}$$
$$=\int_0^\infty exp\{-(n+1)\log(\nu+\sqrt{\nu^2-1}\cosh\phi)\}\,d\phi.$$

Now

$$(n+1)\log\{\nu+\sqrt{\nu^2-1}\cosh\phi\}$$
$$=\tfrac{1}{2}\{1+\sqrt{2\lambda^2c+1}\}\log\left\{1+\frac{\eta^2}{c}+\eta\sqrt{\frac{2}{c}+\frac{\eta^2}{c^2}}\cosh\phi\right\}$$
$$=\lambda\eta\cosh\phi$$

ultimately;

$$\therefore\ Q_n=\int_0^\infty \epsilon^{-\lambda\eta\cosh\phi}\,d\phi=Y_0(\lambda\iota\eta)\ \ldots\ldots\ldots\ldots\ldots(12)$$

ultimately; and is therefore a Bessel's function of the second kind.

In the same way we can show that

$$P_n(\nu)=\frac{1}{\pi}\int_0^\pi \epsilon^{\lambda\eta\cos\phi}\,d\phi \qquad \nu>1$$
$$=J_0(\lambda\iota\eta)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

and that
$$P_n(\mu)=\frac{1}{\pi}\int_0^\pi \epsilon^{\lambda\iota\xi\cos\phi}\,d\phi \qquad \mu<1$$
$$=J_0(\lambda\xi)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

ultimately.

In (12) put $\sinh\phi = z$, then

$$Y_0 = \int_0^\infty \frac{\epsilon^{-\lambda\eta\sqrt{1+z^2}}dz}{\sqrt{1+z^2}}$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\cos\lambda\eta\theta\,.\,d\theta dz}{1+\theta^2+z^2}$$

$$= \int_0^\infty \frac{\cos\lambda\eta\theta\,.\,d\theta}{\sqrt{1+\theta^2}} \quad\text{..........................(15)}$$

$$= \int_0^\infty \cos(\lambda\eta\sinh\chi)\,d\chi.$$

10. The same result may also be arrived at by employing Neumann's transformation of Laplace's equation. For if we put

$$\xi + \iota\eta = \sqrt{\frac{x+\iota y}{c}},$$

then $$\xi = \sqrt{\frac{r}{c}}\cos\frac{\theta}{2}, \quad \eta = \sqrt{\frac{r}{c}}\sin\frac{\theta}{2},$$

and the curves ξ and η are a family of confocal parabolas. Also

$$J^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 = \frac{1}{4cr},$$

$$\therefore \frac{1}{J^2\rho^2} = \frac{1}{\xi^2} + \frac{1}{\eta^2} \quad\text{..........................(16)};$$

hence Neumann's transformation can be employed and Laplace's equation becomes,

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dV}{d\xi}\right) + \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dV}{d\eta}\right) + \left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)\frac{d^2V}{d\theta^2} = 0.$$

Putting as usual $V = U\sin(m\theta + \alpha_m)$, it follows that U can be expressed in a series of products of the form XY, where X is a function of ξ only, which satisfies the equation,

$$\frac{d^2X}{d\xi^2} + \frac{1}{\xi}\frac{dX}{d\xi} + \left(\lambda^2 - \frac{m^2}{\xi^2}\right)X = 0 \quad\text{................(17)}$$

and Y is a function of η only, which satisfies the equation,

$$\frac{d^2Y}{d\eta^2} + \frac{1}{\eta}\frac{dY}{d\eta^2} - \left(\lambda^2 + \frac{m^2}{\eta^2}\right)Y = 0 \quad\text{................(18)}.$$

If the equation of the surface of the conductor is $\eta = 1$, the

roper solution of (17) will be $X = J_m(\lambda\xi)$. The complete solution f (18), expressed in the usual but somewhat awkward notation, is

$$Y = AJ_m(\lambda\iota\eta) + BY_m(\lambda\iota\eta)$$

vhere $\iota = \sqrt{-1}$; German writers also employ the symbol $K_m(\lambda\iota\eta)$ n place of Y_m. But since both these functions may be treated s real quantities, it appears to me that the introduction of an maginary quantity in the argument creates such needless comlexity as to constitute a fatal objection to the use of this notation; nd I shall therefore employ the symbols $I_m(\lambda\eta)$ and $K_m(\lambda\eta)$ in he place of $J_m(\lambda\iota\eta)$ and $Y_m(\lambda\iota\eta)$ respectively. The complete ntegral of (18) may now be written

$$Y = AI_m(\lambda\eta) + BK_m(\lambda\eta),$$

vhere
$$I_m = \frac{(-)^m\lambda^m\eta^m}{2^m\sqrt{\pi}\Gamma(m+\frac{1}{2})}\int_0^\pi \epsilon^{\lambda\eta\cos\phi}\sin^{2m}\phi\, d\phi \ldots\ldots (19),$$

$$K_m = \frac{2^m\Gamma(m+\frac{1}{2})}{\lambda^m\eta^m\sqrt{\pi}}\int_0^\infty \frac{\cos\lambda\eta\theta\,.\,d\theta}{(1+\theta^2)^{\frac{2m+1}{2}}} * \ldots\ldots\ldots\ldots (20).$$

The integrals (19) and (20) can be easily shown to satisfy (18). hey may however be obtained otherwise as follows. Writing x or $\lambda\eta$, the ordinary expression for $J_m(x)$ is

$$\frac{x^m}{2^m\sqrt{\pi}\Gamma(m+\frac{1}{2})}\int_0^\pi \cos(x\cos\phi)\sin^{2m}\phi\, d\phi.$$

Vhence (19) follows at once by changing x into ιx, rejecting the maginary factor, and multiplying the result by $(-)^m$.

A different expression for $I_m(x)$ may be obtained as follows. n (19) put $\lambda\eta = x$, $y = x^m u_m$, then it can be shown that

$$\frac{du_m}{d(x^2)} = u_{m+1},$$

$$\therefore\ I_m = x^m\left[\frac{d}{d(x^2)}\right]^m I_0.$$

Now
$$I_0(x) = \frac{1}{\pi}\int_0^\pi \epsilon^{-x\cos\phi}\, d\phi$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \epsilon^{-x\cos\phi}\, d\phi$$

* This integral has been evaluated by Prof. J. J. Thomson, *Quarterly Journal*, ol. XVIII. p. 377.

$$= \frac{4}{\pi^2} \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{\eta \sin x\eta \, d\eta \, d\phi}{\eta^2 + \cos^2 \phi}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin x\eta \, d\eta}{(\eta^2 + 1)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \chi d\chi}{(x^2 + \chi^2)^{\frac{1}{2}}};$$

$$\therefore I_m = \frac{2(-)^m x^m \Gamma(m + \frac{1}{2})}{\pi^{\frac{3}{2}}} \int_0^\infty \frac{\sin \chi d\chi}{(x^2 + \chi^2)^{\frac{2m+1}{2}}}$$

$$= \frac{2(-)^m \Gamma(m + \frac{1}{2})}{x^m \pi^{\frac{3}{2}}} \int_0^\infty \frac{\sin x\theta \, d\theta}{(1 + \theta^2)^{\frac{2m+1}{2}}}.$$

The value of K_m may be deduced in a similar manner from (15).

Since the functions $P_n^m(\nu)$ and $Q_n^m(\nu)$, where $\nu > 1$, ultimately reduce to the functions I_m and K_m respectively; and the function $P_n^m(\mu)$, where $\mu < 1$, ultimately reduces to the function J_m; it follows that the potential at all points outside the paraboloid $\eta = 1$ is of the form,

$$V = \Sigma \sin(m\phi + \alpha_m) \int_0^\infty F(\lambda) \frac{K_m(\lambda\eta) J_m(\lambda\xi)}{K_m(\lambda)} d\lambda,$$

and at an internal point

$$V' = \Sigma \sin(m\phi + \alpha_m) \int_0 F(\lambda) \frac{I_m(\lambda\eta) J_m(\lambda\xi)}{I_m(\lambda)} d\lambda,$$

where F is a function whose value depends upon the nature of the problem considered.

If we invert these results with respect to the focus, it follows that the potential at all points *outside* the surface bounded by the revolution of the cardioid $\eta = 1$ is

$$V = \frac{c}{r} \Sigma \sin(m\phi + \alpha_m) \int_0^\infty F(\lambda) \frac{I_m(\lambda\eta) J_m(\lambda\xi)}{I_m(\lambda)} d\lambda \ldots (21),$$

and at an internal point,

$$V' = \frac{c}{r} \Sigma \sin(m\phi + \alpha_m) \int_0^\infty F(\lambda) \frac{K_m(\lambda\eta) J_m(\lambda\xi)}{K_m(\lambda)} d\lambda \ldots (22),$$

where in this case,

$$\xi = \sqrt{\frac{c}{r}} \cos\frac{\theta}{2}, \quad \eta = \sqrt{\frac{c}{r}} \sin\frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots (23).$$

11. The properties of the J functions are so fully discussed in odhunter's *Functions of Laplace* that it is unnecessary to consider hem here, further than to note that the principal equations which hey satisfy are the following, viz.

$$\left.\begin{aligned} J_m' &= \frac{m}{x} J_m - J_{m+1} \\ J_m' &= J_{m-1} - \frac{m}{x} J_m \\ J_0' &= -J_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

here x is the argument and the accents denote differentiation.

By means of the definite integrals (19) and (20), it can easily e shown that the I and K functions both satisfy the equations

$$\left.\begin{aligned} I_m' &= \frac{m}{x} I_m - I_{m+1} \\ I_m' &= -\frac{m}{x} I_m - I_{m-1} \\ I_0' &= -I_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(25).$$

12. Again, since I_m and K_m both satisfy the equation

$$I_m'' + \frac{I_m'}{x} - \left(1 + \frac{m^2}{x^2}\right) I_m = 0,$$

 follows that

$$I_m''K_m - I_mK_m'' + \frac{1}{x}(I_m'K_m - I_mK_m') = 0;$$

$$\therefore \quad I_m'K_m - I_mK_m' = \frac{C(-)^m}{x},$$

here C is independent of x. Substituting for I_m', K_m' their values om the first and second of equations (25) respectively, we btain

$$I_{m+1}K_m - I_mK_{m+1} = \frac{C(-)^{m+1}}{x},$$

nd $$I_mK_{m-1} - I_{m-1}K_m = \frac{C(-)^m}{x};$$

hence $$I_1K_0 - I_0K_1 = -\frac{C}{x};$$

herefore $$I_0'K_0 - I_0K_0' = \frac{C}{x}.$$

13. Now when x is a small quantity, $I_0 = 1$, $I_0' = \frac{x}{2}$ approximately; also

$$K_0 = \int_0^\infty \frac{\cos\phi\, d\phi}{(x^2+\phi^2)^{\frac{1}{2}}};$$

$$\therefore\quad K_0' = -\int_0^\infty \frac{x\cos\phi\, d\phi}{(x^2+\phi^2)^{\frac{3}{2}}}$$

$$= -\frac{1}{x}\int_0^\infty \frac{\cos x\theta\, d\theta}{(1+\theta^2)^{\frac{3}{2}}};$$

$$\therefore\quad xK_0' = -1$$

when $x = 0$. Also,

$$x^2K_0 = xK_0' + x^2K_0'',$$

and
$$K_0'' = -\int_0^\infty \frac{\cos\phi\, d\phi}{(x^2+\phi^2)^{\frac{3}{2}}} + 3\int_0^\infty \frac{x^2\cos\phi\, d\phi}{(x^2+\phi^2)^{\frac{5}{2}}};$$

$$\therefore\quad x^2K_0'' = -\int_0^\infty \frac{\cos x\theta\, d\theta}{(1+\theta^2)^{\frac{3}{2}}} + 3\int_0^\infty \frac{\cos x\theta\, d\theta}{(1+\theta^2)^{\frac{5}{2}}}$$

$$= 1$$

when $x = 0$. Therefore $x^2K_0 = 0$ when $x = 0$;

$$\therefore\quad \frac{x^2}{2}K_0 - xK_0' = 1,\ x = 0;$$

$$\therefore\quad C = 1.$$

Hence
$$I_m'K_m - I_mK_m' = \frac{(-)^m}{x} \qquad \text{.................(26).}$$

14. We shall now apply the preceding analysis to determine the potential of the induced charge, when a cardioid of revolution is placed in a field of force whose potential is

$$V_0 + Ax + B\rho\sin\theta.$$

If V, V_n, V_ρ be the respective portions of the resulting potential, we have shown that

$$V = \frac{c}{r}\int_0^\infty F(\lambda)\frac{I_0(\lambda\eta)\,J_0(\lambda\xi)}{I_0(\lambda)}\,d\lambda,$$

where $\eta = 1$ is the equation of the surface of the conductor. Hence at the surface

$$\int_0^\infty F(\lambda)\,J_0(\lambda\xi)\,d\lambda = -\frac{V_0 r}{c}$$

$$= -\frac{V_0}{\xi^2+1} \qquad \text{..........(27).}$$

By means of the theorem*

$$\phi(\xi) = \int_0^\infty \int_0^\infty \lambda\alpha\phi(\alpha)\, J_m(\lambda\alpha)\, J_m(\lambda\xi)\, d\alpha\, d\lambda \quad \text{.........(28)},$$

it follows that

$$\frac{1}{1+\xi^2} = \int_0^\infty \int_0^\infty \frac{\lambda\alpha}{1+\alpha^2} J_0(\lambda\alpha)\, J_0(\lambda\xi)\, d\alpha\, d\lambda.$$

But
$$K_0(\lambda) = \int_0^\infty \frac{\alpha J_0(\lambda\alpha)\, d\alpha}{1+\alpha^2} \dagger \quad \text{..................(29)};$$

$$\therefore \quad \frac{1}{1+\xi^2} = \int_0^\infty \lambda K_0(\lambda) J_0(\lambda\xi)\, d\lambda \quad \text{............(30)};$$

$$\therefore \quad F(\lambda) = -V_0\lambda K_0(\lambda),$$

and
$$V = -\frac{V_0 c}{r} \int_0^\infty \frac{\lambda K_0(\lambda)}{I_0(\lambda)} I_0(\lambda\eta)\, J_0(\lambda\xi)\, d\lambda \quad \text{.........(31)}.$$

Secondly, consider the term Ax; the surface condition is

$$\int_0^\infty F(\lambda)\, J_0(\lambda\xi)\, d\lambda = -\frac{Arx}{c}$$

$$= -Ac \cdot \frac{\xi^2 - 1}{(\xi^2+1)^3}$$

$$= -Ac \int_0^\infty \int_0^\infty \frac{\lambda(\alpha^2-1)\alpha}{(\alpha^2+1)^3} J_0(\lambda\alpha)\, J_0(\lambda\xi)\, d\alpha\, d\lambda.$$

In (29) put $\lambda\alpha = \theta$, then

$$K_0 = \int_0^\infty \frac{\theta J_0(\theta)\, d\theta}{\lambda^2+\theta^2},$$

$$\therefore \quad K_0' = -2\lambda \int_0^\infty \frac{\theta J_0(\theta)\, d\theta}{(\lambda^2+\theta^2)^2} = -\frac{2}{\lambda}\int_0^\infty \frac{\alpha J_0(\lambda\alpha)\, d\alpha}{(1+\alpha^2)^2};$$

$$\therefore \quad K_0'' = \frac{K_0'}{\lambda} + 8\lambda^2 \int_0^\infty \frac{\theta J_0(\theta)\, d\theta}{(\lambda^2+\theta^2)^3};$$

$$\therefore \quad K_0 = \frac{K_0'}{\lambda} + K_0'' = -\frac{4}{\lambda^2}\int_0^\infty \frac{\alpha(\alpha^2-1)\, J_0(\lambda\alpha)\, d\alpha}{(1+\alpha^2)^3} \quad \text{......(31 a)}.$$

$$\therefore \quad \frac{\xi^2-1}{(\xi^2+1)^3} = -\tfrac{1}{4}\int_0^\infty \lambda^3 K_0(\lambda)\, J_0(\lambda\xi)\, d\lambda \quad \text{.............(32)};$$

* See *Proc. Camb. Phil. Soc.*, Vol. v. p. 427.

† Heine *Kugelfunctionen*, Vol. I. p. 197.

$$\therefore \quad F(\lambda) = \frac{Ac}{4}\lambda^3 K_0(\lambda);$$

$$\therefore \qquad = V_x \frac{Ac^2}{4r}\int_0^\infty \frac{\lambda^3 K_0(\lambda)}{I_0(\lambda)} I_0(\lambda\eta) J_0(\lambda\xi)\, d\lambda \text{.....(33).}$$

Lastly, consider the term $B\rho\sin\theta$; the surface condition is

$$\int_0^\infty F(\lambda) J_1(\lambda\xi)\, d\lambda = -\frac{Br\rho}{c}$$

$$= -\frac{2Bc\xi}{(\xi^2+1)^3}$$

$$= -2Bc\int_0^\infty\int_0^\infty \frac{\lambda\alpha^2}{(1+\alpha^2)^3} J_1(\lambda\alpha) J_1(\lambda\xi)\, d\alpha d\lambda.$$

Now we have shown that

$$-K_0'(\lambda) = K_1(\lambda) = 2\lambda\int_0^\infty \frac{\theta J_0(\theta)\, d\theta}{(\lambda^2+\theta^2)^2};$$

also $$\theta J_0 = J_1 + \theta J_1',$$

$$\therefore \quad K_1 = 2\lambda\int_0^\infty (J_1 + \theta J_1')\frac{d\theta}{(\lambda^2+\theta^2)^2}.$$

Integrating the last term by parts, and then putting $\theta = \lambda\alpha$, we obtain

$$K_1(\lambda) = \frac{8}{\lambda^2}\int_0^\infty \frac{\alpha^2 J_1(\lambda\alpha)\, d\alpha}{(1+\alpha^2)^3} \text{.......................(33 a);}$$

$$\therefore \quad \frac{\xi}{(\xi^2+1)^3} = \tfrac{1}{8}\int_0^\infty \lambda^3 K_1(\lambda) J_1(\lambda\xi)\, d\lambda \text{ ....(34);}$$

$$\therefore \quad F(\lambda) = -\frac{Bc}{4}\lambda^3 K_1(\lambda);$$

$$\therefore \qquad V_\rho = -\frac{Bc^2\sin\theta}{4r}\int_0^\infty \frac{\lambda^3 K_1(\lambda)}{I_1(\lambda)} I_1(\lambda\eta) J_1(\lambda\xi)\, d\lambda \text{...(35).}$$

If the solid be moving parallel to its axis with velocity U, the current function

$$\psi = \frac{V_\rho U\rho \operatorname{cosec}\theta}{B} \text{..........................(36),}$$

where value of V_ρ is given by (35).

15. *To find the attraction of a homogeneous cardioid of revolution of unit density.*

Let X, X' be the x-components at an external and internal point respectively, then

$$X = \frac{c}{r}\int_0^\infty F(\lambda)\, K_0(\lambda)\, I_0(\lambda\eta)\, J_0(\lambda\xi)\, d\lambda,$$

$$X' = \frac{c}{r}\int_0^\infty F(\lambda)\, I_0(\lambda)\, K_0(\lambda\eta)\, J_0(\lambda\zeta)\, d\lambda.$$

Also $$dn = -\frac{d\eta}{J} = -\frac{r^{\frac{3}{2}}}{\sqrt{c}}\, d\eta,$$

$$\therefore\ \frac{dX}{d\eta} - \frac{dX'}{d\eta} = \frac{4\pi r^{\frac{3}{2}}}{\sqrt{c}}\, l.$$

By (26) this becomes

$$\int_0^\infty F(\lambda)\, J_0(\lambda\xi)\, d\lambda = \frac{4\pi}{c^{\frac{3}{2}}} r^{\frac{5}{2}} l \qquad \text{.....................(37).}$$

Now if ϕ be the angle which the radius vector drawn from the origin to any point on the curve makes with the axis,

$$l = \sin\frac{3\phi}{2}$$

$$= \frac{3\xi^2 - 1}{\xi^2 + 1}\cdot\sqrt{\frac{r}{c}}.$$

Therefore (37) becomes

$$\int_0^\infty F(\lambda)\, J_0(\lambda\xi)\, d\lambda = \frac{4\pi c\,(3\xi^2 - 1)}{(\xi^2+1)^4}$$

$$= 4\pi c\int_0^\infty\int_0^\infty \frac{\lambda\alpha\,(3\alpha^2 - 1)}{(\alpha^2+1)^4} J_0(\lambda\alpha)\, J_0(\lambda\xi)\, d\alpha\, d\lambda.$$

Let $$u = \int_0^\infty \frac{\alpha\,(3\alpha^2 - 1)}{(\alpha^2+1)^4} J_0(\lambda\alpha)\, d\alpha$$

$$= \int_0^\infty \left\{\frac{3}{(\alpha^2+1)^3} - \frac{4}{(\alpha^2+1)^4}\right\} \alpha J_0(\lambda\alpha)\, d\alpha.$$

Put $\lambda\alpha = \theta$ in (31 a), and differentiating with respect to λ, we obtain

$$K_0'(\lambda) = 16\int_0^\infty \frac{\theta\lambda\,(2\theta^2 - \lambda^2)\, J_0(\theta)\, d\theta}{(\theta^2+\lambda^2)^4}$$

$$= \frac{16}{\lambda^3}\int_0^\infty \frac{\alpha\,(2\alpha^2 - 1)\, J_0(\lambda\alpha)\, d\alpha}{(\alpha^2+1)^4};$$

$$\therefore\ 3u = \frac{\lambda^3 K_0'}{4} + \int_0^\infty \frac{\alpha J_0(\lambda\alpha)\, d\lambda}{(\alpha^2+1)^3}$$

$$= \frac{\lambda^3 K_0'}{4} + \frac{\lambda K_0'}{4} - \frac{\lambda^2 K_0}{8};$$

$$\therefore \quad u = \tfrac{1}{12}\left\{\lambda^3 K_0' + \lambda K_0' - \frac{\lambda^2 K_0}{2}\right\};$$

$$\therefore \; F(\lambda) = \frac{\pi c \lambda^2}{6}\{2(\lambda^2+1) K_0' - \lambda K_0\} \quad \text{..................(38);}$$

whence the attractions can be at once written down.

Secondly, let R, R' be the components perpendicular to the axis, then

$$R = \frac{c}{r}\int_0^\infty F_1(\lambda)\, K_1(\lambda)\, I_1(\lambda\eta)\, J_1(\lambda\xi)\, d\lambda,$$

$$R' = \frac{c}{r}\int_0^\infty F_1(\lambda)\, I_1(\lambda)\, K_1(\lambda\eta)\, J_1(\lambda\xi)\, d\lambda,$$

and at the surface,

$$\frac{dR}{d\eta} - \frac{dR'}{d\eta} = \frac{4\pi r^{\frac{3}{2}}}{\sqrt{c}}\, m.$$

By (26) this becomes

$$\int_0^\infty F(\lambda)\, J_1(\lambda\xi)\, d\lambda = -\frac{4\pi}{c^{\frac{3}{2}}}\, r^{\frac{5}{2}} m \quad \text{..................(39).}$$

Now
$$-m = \cos\frac{3\phi}{2}$$

$$= \xi\,\frac{\xi^2-3}{\xi^2+1}\sqrt{\frac{r}{c}}.$$

Therefore (39) becomes

$$\int_0^\infty F_1(\lambda)\, J_1(\lambda\xi)\, d\lambda = \frac{4\pi c\,(\xi^2-3)\,\xi}{(\xi^2+1)^4}$$

$$= 4\pi c\int_0^\infty\int_0^\infty \frac{\lambda\alpha^2(\alpha^2-3)}{(\alpha^2+1)^4} J_1(\lambda\alpha)\, J_1(\lambda\xi)\, d\alpha\, d\lambda.$$

Let
$$u = \int_0^\infty \frac{\alpha^2(\alpha^2-3)}{(\alpha^2+1)^4} J_1(\lambda\alpha)\, d\alpha$$

$$= \int_0^\infty \left\{\frac{\alpha^2}{(\alpha^2+1)^3} - \frac{4\alpha^2}{(\alpha^2+1)^4}\right\} J_1(\lambda\alpha)\, d\alpha.$$

Now by (33 a)

$$K_1(\lambda) = \frac{8}{\lambda^2}\int_0^\infty \frac{\alpha^2 J_1(\lambda\alpha)\, d\alpha}{(\alpha^2+1)^3}.$$

Let $\lambda\alpha = \theta$, then

$$K_1(\lambda) = 8\lambda\int_0^\infty \frac{\theta^2 J_1(\theta)\, d\theta}{(\lambda^2+\theta^2)^3};$$

$$\therefore\ K_1'(\lambda) = \frac{K_1(\lambda)}{\lambda} - 48\lambda^2 \int_0^\infty \frac{\theta^2 J_1(\theta)\,d\theta}{(\lambda^2+\theta^2)^4}$$

$$= \frac{K_1(\lambda)}{\lambda} - \frac{48}{\lambda^3}\int_0^\infty \frac{\alpha^2 J_1(\lambda\alpha)\,d\alpha}{(1+\alpha^2)^4};$$

$$\therefore\ u = \frac{\lambda^2}{24}(K_1 + 2\lambda K_1');$$

$$\therefore\ F_1(\lambda) = \frac{\pi c}{6}\lambda^3 (K_1 + 2\lambda K_1') \ldots\ldots\ldots\ldots\ldots\ldots(40);$$

whence the attractions are determined.

(2) *An attempt to explain certain geological phenomena by the application to a liquid substratum of Henry's law of the absorption of gases.* By Rev. O. FISHER, M.A.

HENRY'S law of the absorption of gases by liquids asserts, that the volume of the gas which can be held in solution by the liquid is the same, whatever be the pressure. But since a given volume of gas contains, by Boyle's law, a mass proportional to the pressure, it follows, that the mass of a gas that can be held in solution by a given quantity of liquid at a constant temperature varies as the pressure.

Suppose that a liquid substratum exists beneath the earth's crust, and that it consists of fused rock, holding gas (chiefly water above its critical temperature) in solution. It is believed by the writer to be this water which is given off during volcanic eruptions*. If such be the constitution of the substratum, it is obvious that, not only will the reactions between this and the crust largely depend upon it, but also the tidal effects of the moon and sun upon the earth be modified.

Let m be the volume of gas held in solution in unit volume of the liquid, γp the mass of the same under the pressure p; where m and γ are two constants, m depending upon the solubility of the particular gas in the particular liquid at the given temperature, and γ depending upon the compressibility of the gas at that temperature.

If the analogy of water dissolving different gases at ordinary temperatures can be taken as a guide, it would seem that m will vary greatly for different gases, as appears from Henry's experiments. In the case of carbonic acid and water he found m to be about unity. In the case of sulphuretted hydrogen and water, m was about 0·86†.

* Prof. Prestwich refers the presence of steam in volcanic eruptions to the lava encountering water, derived from meteoric or marine sources, during its rise in the vent. *Proc. Roy. Soc.* Apr. 16, 1885.

† *Phil. Trans. Roy. Soc.* 1803.

There is no *a priori* reason why the applicability of the physical law should be confined to the ordinary range of temperature, or why the relations between a substance, as rock, which is not liquid at a lower temperature than (say) 3000° Fah., and a substance, as water, which is not a gas under about 700° Fah., should differ essentially from those between a substance (water) and a gas (e.g. carbonic acid), which are respectively liquid and gaseous at ordinary temperatures. It is quite possible that the solubility of water-gas in molten rock may be as great as that of carbonic acid in water, in which case we should have m about equal to unity. But apart from experiment, there can be no certainty upon this ratio.

If the fused rock is saturated with gas at all depths, the mass of the gas dissolved by a given quantity of it will be greater at greater depths on account of the increased pressure there: and if, on the pressure being relieved from that required for saturation, vesicles of gas are separated from it, these must be subject to the liquid pressure of the magma.

Let the magma be exactly saturated at every depth ζ under the pressure ϖ at that depth. It will consist of fused rocky matter and dissolved gas, without any vesicles of free gas, and if the density of the rocky matter is σ, the density of the magma will be $\sigma + \gamma\varpi$. We may regard the force of gravity (g) as constant for the depths we intend to deal with, and we have to determine ϖ,

$$\varpi = g \int (\sigma + \gamma\varpi)\, d\zeta.$$

Suppose there to be a solid cooled crust of thickness k and density σ. Then

$$\varpi = \left(\frac{\sigma}{\gamma} + g\sigma k\right) \epsilon^{\gamma g \zeta} - \frac{\sigma}{\gamma}.$$

In fact the crust must be of less density than the substratum, being probably about 2·68 (that of granite); while $\sigma = 2{\cdot}96$ (the density of basalt). The false assumption can be corrected by giving a compensatory less value to k.

We can now see what is the order of magnitude of the product $\gamma g \zeta$. For since the density where the pressure is ϖ will be $\sigma + \gamma\varpi$, if we put for ϖ the value just obtained, we find under the pressure of a column of the saturated magma, including the pressure of the crust, at the depth ζ,

$$\gamma g \zeta = \log_\epsilon \frac{\sigma'}{\sigma} - \log_\epsilon (1 + \gamma g k)$$

$$< \log_\epsilon \frac{\sigma'}{\sigma},$$

where $\sigma' - \sigma$ is the increment of density over the surface density, due solely to the absorbed gas. The increment of density in nature is however thought to be partly due to the presence of heavier elements. *A fortiori* then is $\gamma g \zeta$ less than the hyperbolic logarithm of this ratio, such as it actually exists. Prof. Green has calculated the densities from Laplace's formula, using 2·5 as the surface density*. This is probably too small. But since we are concerned with the ratio only, it is near enough for our purpose. He makes the density at 250 miles to be 3·1. Consequently at that depth, regarded as measured from the free surface, as Laplace's law contemplates, we should have $\gamma g(\zeta + k) = 0{\cdot}1914$, which is less than $\frac{1}{5}$th. At a depth of 150 miles, it would be $\frac{3}{5}$ths, or about $\frac{1}{10}$th, and at a depth of 50 miles, it would be $\frac{1}{5}$th of this, or less than $\frac{1}{25}$th.

For the sake of illustration, suppose that, when

$$\zeta + k = 150 \text{ miles} = 6k, \ \gamma g(\zeta + k) = 0{\cdot}115.$$

Then $\gamma g k$ will be 0·019, which implies that the gas dissolved at the bottom of the crust is 0·019 of the mass of the liquid rock holding it in solution. Its mass will therefore be, in a unit mass of rock, $\sigma \times 0{\cdot}019$, say $2{\cdot}68 \times 0{\cdot}019$, or 0·051 of the standard substance, which is liquid water. At this rate, one cubic foot of magma as it exists just beneath the crust, would yield 87 cubic inches of liquid water. For the reason given, this value of $\gamma g(\zeta + k)$ is probably too large and the estimate excessive; nevertheless it suffices to show that the theory will account for the emission of a very considerable quantity of steam from the lava during an eruption.

Suppose that the pressure upon that portion of the magma where it was ϖ becomes p, being less than ϖ. Some of the gas will then be liberated; and by that means let the element $d\zeta$ become dz. The volume of the gas which was wholly dissolved in $d\zeta$ was $md\zeta$. In dz the total volume of the gas, which will be partly free and partly dissolved, being inversely as the pressure will be

$$m\frac{\varpi}{p}d\zeta - md\zeta.$$

This will be the volume by which the element is expanded by change of pressure from ϖ to p: so that

$$dz = d\zeta + m\frac{\varpi}{p}d\zeta - md\zeta.$$

But since the amount of matter in the column is the same identically as before, p can differ from ϖ only by the difference of

* *Geology for Students*, p. 482. 1876.

the surface pressures. Suppose the surface pressure corresponding to p to be $g\sigma c$, we then get the value of p from that of ϖ by subtracting $g\sigma k$ and adding $g\sigma c$ in its place, *i.e.* by subtracting $g\sigma(k-c)$; whence

$$dz = \left\{1 + \frac{mg\sigma(k-c)}{\left(\frac{\sigma}{\gamma} + g\sigma k\right)\epsilon^{\gamma g\zeta} - \frac{\sigma}{\gamma} - g\sigma(k-c)}\right\} d\zeta.$$

Observing that z and ζ begin together, we obtain by integration

$$z = \zeta + \frac{m(k-c)}{1+\gamma g(k-c)} \log_\epsilon \left\{1 + \frac{1+\gamma g(k-c)(1-\epsilon^{-\gamma g\zeta})}{\gamma g c}\right\}.$$

To obtain an approximate expression for the upswelling of the column, suppose that the depth of the substratum is n times that of the crust, or $\zeta = nk$, k being taken at 25 miles*.

Then $\gamma g(\zeta + k) = \gamma g \zeta \frac{n+1}{n} = \lambda$, suppose,

and $$\gamma g \zeta = \frac{n}{n+1}\lambda, \text{ and } \gamma g k = \frac{1}{n+1}\lambda.$$

Substituting these values, we have by expanding and neglecting terms in $\left(\frac{n\lambda}{n+1}\right)^2$,

$$\frac{z-\zeta}{k-c} = m \log_\epsilon \left\{1 + n\frac{k}{c} - \frac{n\lambda}{n+1}\left(\frac{n}{2} - 1\right)\frac{k}{c} - \frac{n\lambda}{n+1} + \text{terms in } \left(\frac{n\lambda}{n+1}\right)^2\right\}\{1 - \gamma g(k-c)\}.$$

The interpretation of this equation will depend upon the value of the ratio $\frac{k}{c}$. Since c is less than k this will be greater than unity. But if it be less by only a few feet, k being taken at say about 25 miles, $\frac{k}{c}$ may be put equal to unity, and then,

$$\frac{z-\zeta}{k-c} = m\left\{\log_\epsilon(1+n) - \frac{1}{2}\left(\frac{n}{1+n}\right)^2\lambda + \frac{k-c}{c}\frac{n\lambda}{(1+n)^2}\right\}\{1-\gamma g(k-c)\}.$$

We have seen that if $\zeta + k$, which is the depth from the surface down to the supposed solid core, is 250 miles $\gamma g(\zeta + k)$ or λ

* See the author's *Physics of the Earth's Crust.*

would be according to Laplace's formula about 0·1914. If the depth were 150 miles, λ would be about 0·115, and n about 5. If we use the latter values, the terms in λ will be less than 0·039, and for our purpose may be neglected, and we arrive at the result,

$$z - \zeta = m \log_\epsilon (1 + n) \times (k - c) \text{ nearly.}$$

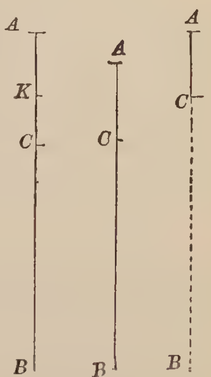
If now we suppose a still further quantity of gas to be liberated owing to the surface pressure being further slightly reduced from $g\sigma c$ to $g\sigma c'$, and that the height of the column of magma then becomes z', we have, by substituting z' for z and subtracting,

$$z' - z = m \log_\epsilon (1 + n)(c - c').$$

The upswelling of the column is consequently sensibly proportional to the diminution of the superincumbent pressure, whether there be already vesicles or not, so long as the rocky matter is saturated with gas.

The doctrine of the entire solidity of the earth rests chiefly upon the fact that no appreciable tides can exist in the interior of the globe, such for instance as would be formed in a substratum of liquid freely intercommunicating throughout a layer of the sphere, provided the liquid were as is usually assumed incompressible. But the constitution of the magma now suggested renders it possible to account for the absence of measureable tides therein.

Let AK be the thickness of the solid crust, KB the depth of the substratum at high tide; and suppose that the rocky matter is completely saturated with gas whether there be free gas among it or not. Let KC be the small space through which the tide might fall, if the column consisted of inexpansible liquid.



(1) (2) (3)

This fall would be caused by the difference of an amount of liquid represented by KC within the column.

Then if we neglect the weight of the small amount of gas in KC, the pressure upon the column CB is lessened by $g\sigma KC$ (see fig. 2), so that on the magma upon that account expanding, we shall have by our formula,

$$\text{expansion of } BC = m \log_\epsilon (1 + n)\, KC.$$

It is obvious that, if the space through which BC expands should be nearly equal to KC there would be no fall of tide observable. This would be the case if $m \log_\epsilon (1 + n)$ were nearly unity, that is, if the volume of gas absorbed according to Henry's law measured

by m, and the depth of the substratum measured by n, were so related that m was the reciprocal of $\log_\epsilon(1+n)$, there would be no tide in the substratum, as for instance if m were about 0·91 and the depth to the solid core 50 miles; or $m = 0·72$ and the depth about 100 miles; or $m = 0·62$ and the depth about 150 miles. These numbers show that, if the solubility of the gas in molten rock was analogous to that of carbonic acid in water or of sulphuretted hydrogen in water; and the depth somewhere between 50 and 100 miles; scarcely any tide would be raised in the substratum, for the effect would consist merely of a change in the vesicularity of the magma following the tide-raising body.

The vesicles of gas liberated during the passage of the body being extremely minute would not have time to rise through the liquid before they were redissolved by the recurring pressure.

It appears therefore that the hypothesis may possibly account for the absence of bodily tides in the earth, and thus remove the chief objection to a liquid substratum.

If from any cause the pressure upon the surface of the substratum were to be permanently lightened, as for instance through denudation or any other cause, the liberated gas would not be redissolved, but the liberated vesicles would coalesce and rise through the magma, expanding as they reached regions of diminished pressure. Thus the expansion of the column would exceed that given by our formulae, which contemplate the vesicles remaining stationary where they are formed. But in spite of the small relative density, it is possible that viscosity would considerably retard the upward movement of the vesicles in their incipient state. It appears therefore that when we come to add the expansion due to the enlargement of the rising vesicles it would exceed the denudation. It seems possible that this may have a bearing upon the direct elevation, without plication, of plateaus like that of Mexico or the Colorado, which is supposed to be connected with volcanic action.

(3) *On a new method of determining Specific Inductive Capacity.* By L. R. WILBERFORCE, B.A.

(*Abstract.*)

The author briefly described the method, which consisted in the comparison of the directive couples upon two spheroids, the one made of the dielectric to be investigated and the other of some conducting material, when they were placed in a uniform electric field. He further indicated certain theoretical considerations with regard to the eccentricities of the spheroids and their manner of suspension, and stated a general theorem relating to the mechanical effect due to such a field upon a body of any material or form.

(4) *On Lagrange's Equations of Motion.* By JAMES C. M^c CONNEL, M.A.

The following proof of Lagrange's equations of motion is founded on that given by Lord Rayleigh in *The Theory of Sound.* It is simplified by supposing the arbitrary displacements of the system not to vary with the time. This alteration brings more clearly into view the part really played by the last term in the final equations of motion. We shall suppose for simplicity that the energy of the system is entirely kinetic.

Let the generalised coordinates be $\psi_1, \psi_2 \ldots$, which are independent, and sufficient to determine the configuration. Let the generalised components of force be $\Psi_1, \Psi_2 \ldots$, and let XYZ be the forces acting on any particle of mass m at the point xyz. Then if ρ_1 be a small arbitrary increment of ψ_1 and $\alpha\beta\gamma$ the corresponding increments of xyz, we have by virtual velocities

$$\Psi_1\rho_1 = \Sigma\,(X\alpha + Y\beta + Z\gamma) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $$\alpha = \frac{dx}{d\psi_1}\rho_1, \quad \beta = \frac{dy}{d\psi_1}\rho_1, \quad \gamma = \frac{dz}{d\psi_1}\rho_1;$$

$$\therefore\ \Psi_1 = \Sigma\left(X\frac{dx}{d\psi_1} + Y\frac{dy}{d\psi_1} + Z\frac{dz}{d\psi_1}\right) \;\ldots\ldots\ldots\ldots(2).$$

This equation is a simple statement of the process by which we can obtain Ψ_1 from the X, Y and Z of all the particles.

Now since

$$\dot{x} = \frac{dx}{d\psi_1}\dot{\psi}_1 + \frac{dx}{d\psi_2}\dot{\psi}_2 + \ldots$$

$$\frac{d\dot{x}}{d\dot{\psi}_1} = \frac{dx}{d\psi_1};$$

$$\therefore\ \frac{dT}{d\dot{\psi}_1} = \Sigma\left(m\dot{x}\frac{d\dot{x}}{d\dot{\psi}_1} + m\dot{y}\frac{d\dot{y}}{d\dot{\psi}_1} + m\dot{z}\frac{d\dot{z}}{d\dot{\psi}_1}\right)$$

$$= \Sigma\left(m\dot{x}\frac{dx}{d\psi_1} + m\dot{y}\frac{dy}{d\psi_1} + m\dot{z}\frac{dz}{d\psi_1}\right) \;\ldots\ldots\ldots\ldots(3).$$

Thus $\dfrac{dT}{d\dot{\psi}_1}$ can be obtained from the momenta of the particles by exactly the same process as that by which Ψ_1 was obtained from the forces acting on the particles.

Multiply both sides by ρ_1 and differentiate with regard to t, remembering that $\alpha = \frac{dx}{d\psi_1}\rho_1$ &c.,

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\rho_1\right) = \Sigma\left(m\ddot{x}\frac{dx}{d\psi_1} + m\ddot{y}\frac{dy}{d\psi_1} + m\ddot{z}\frac{dz}{d\psi_1}\right)\rho_1$$
$$+ \Sigma\,(m\dot{x}\dot{\alpha} + m\dot{y}\dot{\beta} + m\dot{z}\dot{\gamma}).$$

Now ρ_1 is perfectly arbitrary so we shall suppose it not to change with the time, so that the increment of $\dot{\psi}_1$ is zero, but $\alpha\beta\gamma$ are bound by the connections of the system and therefore vary with the time; $\dot{\alpha}$ is the change of $\dot{x}$ produced by ψ_1 being changed to $\psi_1 + \rho_1$, while $\dot{\psi}_1, \dot{\psi}_2 \dots$ as well as $\psi_2, \psi_3 \dots$ are unchanged,

$$\text{i.e. } \dot{\alpha} = \frac{d\dot{x}}{d\psi_1}\rho_1 \text{ and } \dot{\beta} = \frac{d\dot{y}}{d\psi_1}\rho_1,\ \dot{\gamma} = \frac{d\dot{z}}{d\psi_1}\rho_1,$$

$$\therefore\ \frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right)\rho_1 - \frac{dT}{d\psi_1}\rho_1$$
$$= \Sigma\left(m\ddot{x}\frac{dx}{d\psi_1} + m\ddot{y}\frac{dy}{d\psi_1} + m\ddot{z}\frac{dz}{d\psi_1}\right)\rho_1 \quad \dots\dots(4).$$

Thus $\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right) - \frac{dT}{d\psi_1}$ is obtained from the rates of change of momentum of the particles by the same process as in the previous cases, and might be called the generalised component of rate of change of momentum.

It differs from the rate of change of the generalised component of momentum by the term $-\frac{dT}{d\psi_1}$. And this term is entirely due to the circumstance that the coefficients $\frac{dx}{d\psi_1}$ &c. vary with the time.

To complete the proof we need merely remark that by the second law of motion

$$X = m\ddot{x}, \quad Y = m\ddot{y}, \quad Z = m\ddot{z},$$

so that by (2) and (4)

$$\Psi_1 = \frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right) - \frac{dT}{d\psi_1} \quad \dots\dots\dots\dots\dots\dots\dots(5).$$

Clerk Maxwell gives a proof of Lagrange's equations in his *Electricity and Magnetism,* in which no direct reference is made to

[t]he second law of motion. But on examination this proof turns [o]ut to be quite invalid. He obtains* an equation of the form

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right)\rho_1 - \frac{dT}{d\psi_1}\rho_1 + \frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_2}\right)\rho_2 - \frac{dT}{d\psi_2}\rho_2 + \ldots = \Psi_1\rho_1 + \Psi_2\rho_2 + \ldots,$$

[w]hich is established by the remark that the change of kinetic [e]nergy must be equal to the work done by the forces.

Moreover he says that $\rho_1 = \dot{\psi}_1 \delta t$; from both of which state[m]ents it is evident that $\rho_1, \rho_2 \ldots$ are the displacements which [a]ctually occur under the action of the forces $\Psi_1, \Psi_2 \ldots$ in time δt.

So the ρ's are not independent of the Ψ's nor of the $\dot{\psi}$'s, he [c]annot equate all the ρ's to zero except ρ_1, and he is not justified [i]n deducing that the coefficient of ρ_1 on the one side is equal to that [o]n the other.

November 8, 1886.

MR TROTTER, PRESIDENT, IN THE CHAIR.

Mr W. F. Sheppard, B.A., Trinity College, and Mr F. H. [N]eville, M.A., Sidney College, were elected Fellows.

Mr H. F. Reid was elected an Associate.

The following communications were made to the Society.

(1) *On the Cœlom and body-cavity of Peripatus and the Arthro[p]oda.* By ADAM SEDGWICK, M.A., Trinity College.

IT is well known that the vascular system of the Arthropoda is [i]n direct communication with the body-cavity, and that the vessels [a]re, for the most part, very rudimentary. In fact the blood is [d]riven by the heart or dorsal vessel into the body-cavity and [r]eturned directly through the lateral cardiac ostia into the heart. [I]n no other group of animals, so far as I know, does this direct [c]ommunication exist between the heart and the pericardium†.

* § 561, equations (7) and (8).

† This important Arthropodan character has so far escaped the notice of mor[p]hologists: at least I have never seen it mentioned in any of the books. My [a]ttention was first called to it by Professor Ray Lankester.

It is therefore important to determine by the study of development, whether or no the blood-containing body and pericardial cavities of the Arthropoda are homologous with the corresponding structures of other types, in which they do not contain blood.

The development of the Arthropodan heart and body-cavity is in most cases extremely difficult to follow on account of the large amount of food yolk present in the embryos, and we have not, at present, any completely satisfactory history of it.

The development of *Peripatus Capensis*, which is a true Arthropod, so far as its body-cavity and vascular system are concerned, is comparatively easy to follow, and I have been able to make out with precision the history of the parts in question.

The cœlom appears in the ordinary manner as a series of cavities, one in each mesoblastic somite.

The somites, which are at first ventro-lateral in position, soon acquire a dorsal extension and the cavity in each of them becomes divided into two parts,—a ventral part which passes into the appendage, and a dorsal part which comes into contact but does not unite with its fellow of the opposite side on the dorsal wall of the enteron.

The **dorsal portions** of the somites early become obliterated in the anterior part of the body, but posteriorly they persist, and those of the same side unite with each other so as to form two tubes which are the generative glands.

The **ventral** or **appendicular portions** persist and retain their original isolation throughout life. They give rise to two structures—

(1) to a coiled tube, which acquires an external opening through the ventral body wall at the base of the appendage and constitutes the **nephridium** of the adult;

(2) to a small **vesicle** which is contained in the appendage and constitutes the internal blind end of the tubular or nephridial portion of the somite. (The opening of the nephridium into the vesicle is funnel-shaped and is commonly known as the internal funnel-shaped opening of the former.)

From the above account it follows (1) that the cœlom of the embryo of *Peripatus Capensis* gives rise to the nephridia and generative glands, but to no part of the body-cavity of the adult; (2) that the nephridia of the adult do not open into the body-cavity.

The **body-cavity** of the adult consists as is well known of four divisions: (*a*) the central compartment containing the intestine and generative organs, (*b*) the pericardial cavity, (*c*) the lateral compartments containing the nerve cords and salivary glands, and (*d*) the portion in the appendage.

Of these, without going into details, it may be said that

arises as a space between the ectoderm and the endoderm, *b*, *c*, and as spaces in the thickened somatic walls of the somites. The spaces are in communication with each other.

The **heart** arises as a part of *a* which becomes separated from the rest. Posteriorly it acquires paired openings into the pericardium. It thus appears that the heart and various divisions of the body-cavity of the adult form a series of spaces which have nothing to do with the cœlom. They all communicate with each other and seem to form a series of enormously dilated vascular trunks, of which the heart is the narrowest and alone possesses the property of rhythmically contracting.

To sum up it appears that the cœlom in Peripatus is an inconspicuous structure in the adult, and has no connection with the body-cavity; while, on the other hand, the spaces of the vascular system are but little subdivided, and form the heart and various divisions of the adult body-cavity.

If these results are applicable to the Arthropoda generally, and I see no reason, from the similarity of the adult anatomy, to doubt that they will be found to be so, we may add the following morphological features to those generally stated as appertaining to the group: *cœlom inconspicuous, body-cavity consisting entirely of vascular spaces.*

In Vertebrates and most Annelids, on the other hand, the parts in question are arranged as follows: *Body-cavity entirely cœlomic; vascular spaces broken up into a complicated system of channels (arteries, veins, capillaries).*

In most Molluscs, finally, the pericardium alone is cœlomic*; the vascular spaces being represented by the heart and the more or less complicated system of spaces in the body.

In conclusion I may point out that Kennel, who has correctly described the origin of the body-cavity, has failed to elucidate the history of the cœlom in Peripatus. He has also failed to recognise the fact that the nephridia do not open into the body-cavity.

A detailed account of my observations, fully illustrated, is now in course of preparation, and will shortly appear in the *Quarterly Journal of Microscopical Science.*

(2) *Note on the 'Vesicular Vessels' of the Onion.* By S. H. Vines, M.A., Christ's, and A. B. Rendle, St John's.

In investigating the vesicular vessels with the object of determining whether or not the transverse walls are perforated so as to place the cavities of successive segments in communication, the

* I infer this from the observations of Ziegler (*Zeit. f. wiss. Zool.* 41) and others, who have definitely traced back the pericardium to a space in the mesoblastic bands.

authors observed that, in the quiescent winter condition of the bulb, there are patches of callus—easily made conspicuous by staining with corallin—on the transverse walls. From this they infer that the transverse walls are perforated, the canals through them being open in the active, and closed by callus in the quiescent condition of the bulb, just as is the case with sieve-tubes. This inference has, however, to be confirmed by an investigation of the bulb in the active condition. The authors also observed that each segment of a vesicular vessel contains a large nucleus.

(3) *On Epiclemmydia Lusitanica a new genus of algæ.* By M. C. Potter, M. A.

In the southern parts of Spain and Portugal, where rain falls generally only during the winter and spring months, numerous pools remain throughout the summer in the beds of the dried-up streams. To these pools collect during the summer numerous water loving animals among which we find water Tortoises. These Tortoises remain a great deal in the water and their shells on this account afford a suitable nidus for algæ.

The alga, which is the subject of this paper, appears to the naked eye as roundish patches of a deep green colour and occurs generally on the back, but sometimes on the under surface of the Tortoise near its margin.

On cutting sections of one of these patches in a direction perpendicular to the surface of the animal's back, we find the alga to consist of a number of squarish cells, some of which are closely applied to the surface of the Tortoise shell and exposed to the action of the water in which the Tortoise lives, and others forming wedge-shaped masses which penetrate into the Tortoise shell.

The cells of the alga divide in planes parallel and perpendicular to the back of the Tortoise; and by these divisions new cells are continually being formed, but the mass of cells where exposed is never more than a few layers thick, the outermost layer continually forming zoospores or being destroyed. These cells are closely applied to the surface of Tortoise shell and are continually trying to penetrate into it. The algal cells are only able to penetrate into the Tortoise shell by means of any cracks which may occur; so that whenever an algal cell reaches or finds a crack it immediately grows into the crack. At first when the crack is small only a small piece of algal cell can penetrate; this small piece however grows and spreads parallel to the surface of the Tortoise, and divides into cells by planes perpendicular, and when

n this manner the algal cells have become sufficiently large, they ivide by planes parallel to the surface of the Tortoise. By this rocess the wedge-shaped masses of algal cells, which in section are een penetrating the Tortoise shell, are formed and the patches of lgæ increase in size parallel to the surface of the Tortoise, as the ell at the thin end penetrates more and more into the Tortoise hell, the cells at the opposite end continually divide until at last t the thick end of the wedge the outermost portion of Tortoise hell is flaked off and new algal cells are exposed to the action of he water. (Compare Cunningham on Mycoidea parasitica, *Trans. inn. Soc.* 1879.)

If sections of the alga cut as described above are allowed to emain in water the cells which are uninjured will grow and ivide. Those which form the innermost layer, since now the ressure from other cells is relieved, grow and form filaments, he corner cell of a wedge of cells which is penetrating into the ortoise shell will also form a filament. These filaments grow ery irregularly and to some considerable length.

The alga propagates itself by means of zoospores; these are ormed from the layer of cells in contact with the water. The cell bout to form zoospores becomes flask-shaped through a kind f neck being formed, at the same time the cell-contents are ivided up into a number of zoospores which are dehisced through n opening in the neck. The zoospores have the ordinary pyriorm shape with cilia arising from the clear pointed end. They re all exactly alike and of one kind, and swim about for a coniderable time, after which they come to rest and germinate.

Until the complete life-history of Epiclemmydia is known, its xact systematic position cannot be determined; however, from its reen colour it belongs to the *Chlorophyceæ*, and most likely to he sub-order *Confervoideæ*.

My best thanks are due to Professor Moseley, who in the first nstance drew my attention to this alga; to Dr H. Gadow, who iscovered this alga in Portugal; to the Worts' Travelling Scholars' und, by whose assistance I was enabled to conduct my researches n Portugal, and to my friend, Thomas Warden, Esq., of the mines f São Domingos, in whose house I conducted my researches.

(4) *On a peculiar organ of Hodgsonia heteroclita.* By VALTER GARDINER, M.A.

The author gave some account of the gland-bearing organs hich are found in *Hodgsonia:* one in the axil of each of the oliage leaves. A study of the development of these organs emonstrates that they are peculiarly modified leaves, or rather

bracts, since they are associated with the rudimentary flower bud. They are doubtless identical with the similarly modified bracts which occur in connection with the fully developed flowers. The glands are found on the lower surface of the bract and belong to the same type as those of *Luffa,* although of a distinctly higher order. Glands of a similar nature also occur on the under surface of the foliage leaves and on the sepals. The substance secreted by the glands is most probably of the nature of nectar, and the whole structures are to be regarded as extra-floral nectaries. Having shortly described their histology, the author proceeded to make some remarks upon their function. A careful survey of the various gland-bearing genera of the *Cucurbitaceae* and *Passifloraceae,* and a comparison of such cases as those presented by *Passiflora quadrangularis* and *Passiflora foetida,* placed it, in his opinion, beyond doubt that the function of the extra-floral nectaries of the two orders is to attract certain insects—probably ants—which are of service to the plant in protecting it from the attacks of other and harmful insects, such as caterpillars, which are accustomed to creep up the narrow stem for the purpose of devouring or otherwise injuring the young growing shoots.

As regards the fertilisation of *Hodgsonia,* the author showed that there were special contrivances to prevent the animal which fed upon the nectar of the flower from obtaining that of the extra-floral nectaries, and *vice versa,* and stated that considering all the circumstances of the case it was exceedingly probable that fertilisation was accomplished through the agency of a large night-flying moth.

November 22, 1886.

MR TROTTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society.

(1) *Note de Géométrie cinématique.* Par le Professeur A. Mannheim.

Dans son *Traité des fluxions,* Maclaurin a fait connaître une élégante construction du centre de courbure de la développée d'une ellipse. Je retrouverai cette construction à la fin de ce travail en la déduisant d'une autre à laquelle j'arrive par l'application de quelques propositions de *Géométrie cinématique.*

La génération particulière de l'ellipse, dont je fais usage, peut provenir de l'étude de l'hyperboloïde articulé. C'est ce que je vais d'abord montrer : je rattacherai ainsi cette note à ma théorie de l'hyperboloïde articulé* dont elle formera un complément.

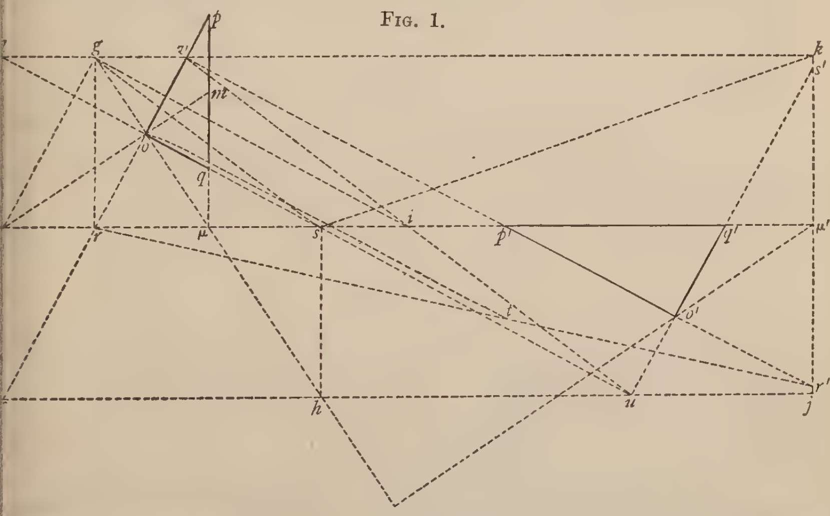
Prenons un prisme droit rectangulaire et ses axes. Sur chacune des faces de ce prisme, parallèle à l'un de ces axes, traçons une diagonale de façon que ces droites se rencontrent successivement et forment un quadrilatère gauche.

On peut déformer ce quadrilatère gauche de manière que deux côtés opposés tournent en sens inverses d'angles égaux en rencontrant toujours à angle droit l'axe auquel ces droites sont perpendiculaires.

Prenons une génératrice de l'hyperboloïde qui contient le quadrilatère gauche et dont les axes sont les axes du prisme. Le segment de cette droite, compris entre deux des côtés opposés du quadrilatère dont je viens de parler, a son point milieu sur l'un des plans principaux de l'hyperboloïde. Pendant la déformation du quadrilatère ce point milieu décrit sur ce plan principal une ellipse, comme je vais le faire voir.

Projetons sur ce plan principal les deux côtés opposés du quadrilatère et le segment de génératrice compris entre ces droites. Limitons les projections ainsi obtenues à leurs points de rencontre ; o étant le centre de l'hyperboloïde ; on obtient ainsi (fig. 1)

FIG. 1.



les droites op, oq, pq et le point m milieu de pq. Pendant la déformation les segments op, oq restent de grandeurs constantes

* *Comptes rendus de l'Académie des Sciences.* Séances des 1er, 8, 15 février et 1er mars 1886.

comme les segments dont ils sont les projections. La déformation du quadrilatère conduit alors à une courbe ainsi engendrée.

Les segments op, oq *de grandeurs constantes tournent, sur le plan de la figure, autour du point* o *d'angles égaux et en sens inverses; l'on prend le lieu des milieux des segments tels que* pq*: je dis que ce lieu est une ellipse.*

Menons les bissectrices ox, oy des angles formés par op et oq. Ces droites restent immobiles pendant la déformation de la figure. Du point q abaissons des perpendiculaires sur ces bissectrices; les portions de ces perpendiculaires comprises entre op et le point q ont leurs points milieux sur ox et oy. La droite qui joint ces points milieux passe par le point m et les distances de ces points au point m sont respectivement égales, comme on le voit sur la figure, à
$$\frac{op+oq}{2} \text{ et } \frac{op-oq}{2}.$$

La courbe décrite par le point m *pendant la déformation de la figure est donc une ellipse qui a pour axes* ox, oy *et dont les longueurs des demi-axes sont* $\frac{\mathrm{op}+\mathrm{oq}}{2}$ *et* $\frac{\mathrm{op}-\mathrm{oq}}{2}$*; la distance du point* o *aux foyers de cette courbe est alors égale à* $\sqrt{\mathrm{op}\times\mathrm{oq}}$.

Pour un déplacement infiniment petit de la droite mobile de grandeur constante dont le point m décrit l'ellipse (m) le point q est le centre instantané de rotation. La droite mq est donc normale à l'ellipse; et si nous revenons à la figure de l'espace, nous voyons que *la droite, dont* pq *est la projection, est toujours normale à cette ellipse pendant la déformation du quadrilatère.*

Démontrons directement que *la droite* pq *est normale en* m *à la courbe lieu des milieux des segments tels que* pq *qui joignent les extrémités des segments mobiles* op, oq*.

Les segments op, oq, tournant d'angles égaux on a:
$$\frac{d(p)}{d(q)}=\frac{op}{oq}.$$

Appelons μ le point où pq touche son enveloppe, élevons de ce point la perpendiculaire rs à pq, on a aussi $\frac{d(p)}{d(q)}=\frac{rp}{sq}$,

et alors
$$\frac{op}{oq}=\frac{rp}{sq}, \text{ ou } \frac{op}{rp}=\frac{oq}{sq}.$$

Projetant sur rs les segments qui entrent dans cette dernière égalité on voit que *le point* μ *est le milieu de* rs.

Le segment pq, limité aux circonférences (p) et (q), touche son

* Je vais employer mes notations habituelles: la ligne décrite par un point tel que p est indiquée par (p) et un arc infiniment petit de cette courbe par $d(p)$.

enveloppe au point μ. Il est partagé par la courbe (m) en deux parties égales : on a alors la normale à (m) en joignant le point m au milieu du segment rs : donc *la droite* pq *est la normale en* m *à la courbe* (m).

De là, nous pouvons aussi déduire que (m) *est une ellipse.* Fixons les droites op et oq et prenons des droites qui les rencontrent en des points tels que le produit de leurs distances à o soit égal à $op \times oq$.

Toutes ces droites enveloppent une hyperbole tangente à pq au point m et qui par suite rencontre alors (m) à angle droit.

Pour une autre position de op et oq on a une nouvelle hyperbole qui rencontre de même (m) à angle droit. Toutes ces hyperboles sont homofocales, car la distance du point o à leurs foyers est égale à $\sqrt{op \times oq}$. *La courbe* (m), *trajectoire orthogonale de ces hyperboles est alors une ellipse homofocale à ces courbes.*

On peut remarquer que le point μ où pq touche son enveloppe est le centre de courbure de (m). D'après ce qui précède on construit μ de la manière suivante : on élève au point q une perpendiculaire à qm. Cette perpendiculaire rencontre en un certain point la droite mobile de grandeur constante dont le point m décrit l'ellipse ; la droite qui joint le point o à ce point coupe mq au centre de courbure μ.

Pendant la déformation de la figure les normales à l'ellipse (m), telle que pq, sont partagées par la circonférence (p), l'ellipse (m) et la circonférence (q) en segments pm et mq qui sont égaux ; de là la possibilité, comme nous venons de le voir, d'obtenir le centre de courbure μ. On pourra de même déterminer le point où rs touche son enveloppe si on peut obtenir les normales aux courbes (r) et (s), car le segment rs normal à la développée de l'ellipse (m) est partagé par cette développée, par (r) et (s) en segments μr et μs qui sont égaux.

Le point où rs touche son enveloppe est le centre de courbure de la développée de l'ellipse (m). C'est ce point que nous allons construire.

Construire le centre de courbure de la développée de l'ellipse (m).

D'après ce que nous venons de dire nous devons d'abord chercher la normale au point r à la courbe (r) décrite par ce point pendant la déformation de la figure. Conservons les notations précédentes et prenons (fig. 2) le triangle μpr qui se déforme en même temps que la figure formée par op et oq ; on a, en appelant μ' le centre de courbure de la développée de (m)

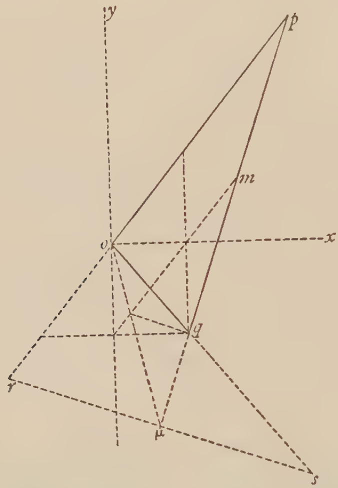
$$\frac{d(\mu)}{d(p)} = \frac{\mu'\mu}{rp},$$

de même, en élevant au point o une perpendiculaire à op et

appelant t le point de rencontre de cette droite et de la normale rr' que l'on cherche, on a

$$\frac{d(p)}{d(r)} = \frac{op}{tr}.$$

Fig. 2.



Enfin le troisième côté μr du triangle $r\mu p$ donne

$$\frac{d(r)}{d(\mu)} = \frac{r'r}{\mu'\mu},$$

multipliant membre à membre ces trois égalités, il vient :

$$1 = \frac{op \times r'r}{rp \times tr}.$$

Abaissons du point r' la perpendiculaire $r'v$ sur op et remplaçons dans cette égalité le rapport $\frac{r'r}{tr}$ par $\frac{vr}{or}$ il vient :

$$\frac{op}{rp} = \frac{or}{vr}.$$

D'après cela on construit ainsi le point v : on élève à rs la perpendiculaire rg, du point g, où cette droite rencontre $o\mu$, on mène une parallèle à rs ; cette parallèle coupe op au point v.

De même, la recherche de la normale ss' conduit à un point u dont la construction est analogue à celle que nous venons de trouver pour v.

Les points r' et s' sont sur la perpendiculaire à rs élevés du point μ' et comme $r\mu = \mu s$ on a aussi $r'\mu' = \mu' s'$. On obtiendra donc le point μ' en cherchant la perpendiculaire à rs dont le seg-

ment compris entre rr' et ss' est partagé par rs en deux parties égales, et en prenant sur rs le pied de cette perpendiculaire.

Il est facile de trouver ce point μ'. En effet les perpendiculaires vr', us', qui se coupent en o', déterminent avec rs un triangle $q'op'$ semblable au triangle qop; la perpendiculaire $o'\mu'$ à $o\mu$ rencontre alors $p'q'$ au point μ' homologue de μ. Ce point μ' est le point cherché: c'est le centre de courbure de la développée de l'ellipse (m).

Comme on le voit *la construction du point* μ' *se réduit à déterminer* v *et* u *et à élever de ces points les perpendiculaires* vr′ *et* us′ *à* op *et* oq*; ces perpendiculaires se coupent au point* o′ *d'où l'on abaisse une perpendiculaire sur* oμ *: cette droite coupe* rs *au point* μ' *cherché.*

Il s'agit maintenant de montrer comment de cette construction on déduit celle qui est due à Maclaurin.

Appelons e le point où le diamètre mo de l'ellipse (m) rencontre rs. Il s'agit de faire voir que $\mu\mu' = 3\mu e$.

Soit l le point où oq rencontre vg, on a $lg = vg$. Par suite la parallèle menée de g à op et la perpendiculaire lf abaissée de l sur rs se coupent en un point de cette dernière droite et ce point est le milieu du segment lf intercepté par op et oq sur cette perpendiculaire. Ce point est alors le point e où le diamètre mo rencontre rs.

Les droites vu et rs se coupent en i qui est le milieu de vu, la droite gi est alors parallèle à oq et comme ge est parallèle à op, on voit que : $\mu i = \mu e$. Il suffit donc de démontrer que

$$i\mu' = ie.$$

Les triangles $us'j$ et luf sont semblables ; ils donnent

$$fu \times uj = s'j \times fl.$$

De même, les triangles semblables vkr' et vlf donnent

$$lv \times vk = kr' \times fl.$$

Comme $s'j$ est égal à kr' on a :

$$fu \times uj = lv \times vk,$$

par suite $lv = uj$ et alors le point i est bien le milieu de $e\mu'$. C'est ce qu'il restait à faire voir pour retrouver l'élégant résultat dû à Maclaurin.

(2) *On the Mechanical Force acting on an Element of a Magnet carrying a Current.* By JAMES C. M^cCONNEL, M.A.

As a foundation for his theory of stresses in the Magnetic Field (*Electricity and Magnetism*, II. p. 254) Maxwell states that the resultant action on a magnetized element carrying a current consists of a force of translation one of whose components is

$$X = A\frac{d\alpha}{dx} + B\frac{d\beta}{dx} + C\frac{d\gamma}{dx} + vc - wb, \qquad (1)$$

and a couple one of whose components is

$$L = B\gamma - C\beta \qquad (2).$$

These expressions are obtained by combining the forces on the two poles of the element.

The notation is that used by Maxwell.

Nowhere in his book does he give any complete proof of this result, and the object of the following paper is to supply one.

It is easy to show that, if a magnetised element, the components of whose magnetisation are A, B, C, be placed in a field of magnetic force whose undisturbed values are α, β, γ, it is subject to a force of translation of which one component is

$$X = A\frac{d\alpha}{dx} + B\frac{d\alpha}{dy} + C\frac{d\alpha}{dz}, \qquad (3)$$

and a couple of which one component is

$$L = B\gamma - C\beta \qquad (4).$$

Next consider an element which forms part of a finite magnet. The magnetic force $\mathfrak{H}$ can now be no longer defined as the force on an unit magnetic pole, but requires a special definition. We shall take that used by Maxwell, which may be thus stated:—

It is zero at an infinite distance from the system of magnets and currents, and satisfies the following equations at every point (§ 607):

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u \qquad (5)$$

with two similar ones,

and (§ 403)
$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = -4\pi\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) \qquad (6).$$

There is only one value of $\mathfrak{H}$ which can satisfy these conditions. $\mathfrak{B}$ is given by the definition

$$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{J}.$$

At present we suppose that there is no current through the element.

Let the element be spherical. We may consider its magnetisation to be uniform and equal to that at its centre without error; for the forces on it, depending on the outstanding irregularities of magnetisation, must be indefinitely small relatively to those depending on the uniform magnetisation. The $\mathfrak{H}$ at any point may be divided into two parts, $\mathfrak{H}_1$ due to the element, and $\mathfrak{H}_2$ due to the rest of the magnet and all external magnets and

currents. Then the element is suspended in a field of force $\mathfrak{H}_2$, so the forces on it are expressed by (3) and (4), with $\alpha_2\beta_2\gamma_2$ written for $\alpha\beta\gamma$.

Now we know that $\mathfrak{H}_1$ within the sphere is uniform and parallel to the magnetisation, so both the force (3) and the couple (4) vanish when $\alpha_1\beta_1\gamma_1$ are written for $\alpha\beta\gamma$. Hence (3) and (4), as they stand, correctly express the forces on the element. Moreover it is obvious that the actual forces must be proportional to the volume and independent of the form of the element.

Now let us consider the general case when a magnetised element is traversed by a current. In this case let the element be a long thin circular cylinder whose length is parallel to the current. By similar arguments to those used before, we may consider the current and the magnetisation to be uniform. We shall avail ourselves of the following results, which are well known or may be easily proved. In an infinite circular cylinder uniformly and transversely magnetised,

$$\mathfrak{H} = -2\pi\mathfrak{I} \quad \text{..............................(7).}$$

In an infinite circular cylinder, uniformly magnetised longitudinally, $\mathfrak{H}$ has no transverse component. In an infinitely long circular cylinder, traversed longitudinally by an uniform current,

$$\left.\begin{aligned} \frac{d\gamma}{dy} &= -\frac{d\beta}{dz} = 2\pi u, \ \&\text{c.} \\ \frac{d\alpha}{dx} &= \frac{d\beta}{dy} = \frac{d\gamma}{dz} = 0 \end{aligned}\right\} \quad \text{..................(8).}$$

In our cylindrical element now let $\mathfrak{H}$ be divided into four parts,

$\mathfrak{H}_1$ due to the current in the element,

$\mathfrak{H}_2$ due to the transverse magnetisation in the element,

$\mathfrak{H}_3$ due to the longitudinal magnetisation in the element,

$\mathfrak{H}_4$ due to the external currents and magnetisation.

It is proved by Maxwell (Art. 490) that the force on the element from outside due to the presence of its current has the components

$$X = v\gamma_4 - w\beta_4, \ \&\text{c.} \quad \text{.........................(9),}$$

the force due to the presence of its magnetisation has the components

$$X = A\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dx} + B\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dy} + C\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dz} \ \&\text{c.} \ \text{...(10),}$$

by the result last obtained.

We may put (10) into the form

$$A\frac{d(\alpha-\alpha_1)}{dx}+B\frac{d(\alpha-\alpha_1)}{dy}+C\frac{d(\alpha-\alpha_1)}{dz}.$$

Now $$\frac{d\alpha_1}{dx}=0,\ \frac{d\beta_1}{dx}=2\pi w,\ \frac{d\gamma_1}{dx}=-2\pi v, \text{ by (8)},$$

and $$\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dy}=\frac{d(\beta_2+\beta_3+\beta_4)}{dx}$$

by the definition of $\mathfrak{H}$,

$$=\frac{d(\beta-\beta_1)}{dx}=\frac{d\beta}{dx}-2\pi w.$$

We can treat the third term of (10) in the same way, so (10) is equal to

$$A\frac{d\alpha}{dx}+B\frac{d\beta}{dx}+C\frac{d\gamma}{dx}-B2\pi w+C2\pi v\text{.........(11)}.$$

Let us now consider (9).

$\mathfrak{H}_1$ is indefinitely small throughout the element,

$\mathfrak{H}_2=-2\pi\mathfrak{J}_2$, if $\mathfrak{J}_2$ be the transverse part of the magnetisation,

$\mathfrak{H}_3$ is parallel to the current, so that $v\gamma_3-w\beta_3$ is zero;

therefore (9) $$=v\gamma-w\beta+(v2\pi C_2-w2\pi B_2)$$
$$=v\gamma-w\beta+\ v2\pi C\ -w2\pi B,$$

since the rest of the magnetisation is parallel to the current.

Adding (11) we find that the total force has the components

$$X=A\frac{d\alpha}{dx}+B\frac{d\beta}{dx}+C\frac{d\gamma}{dx}+v(\gamma+4\pi C)-w(\beta+4\pi B), \text{ \&c.}$$

The couple has the components

$$L=B(\gamma_2+\gamma_3+\gamma_4)-C(\beta_2+\beta_3+\beta_4)$$
$$=B\gamma-C\beta, \text{ \&c.}$$

and these are Maxwell's expressions for the total force and couple.

This completes the proof; but there is one point of considerable interest, which occurs in the course of it, and demands some further remarks. In Art. 490, as we have said, Maxwell shows that the mechanical force acting on a conductor is $V.\mathfrak{C}\mathfrak{H}_4$, using Quaternion notation. Where there is no magnetisation this is the same as $V.\mathfrak{C}\mathfrak{B}$, and Maxwell adopts this latter form and retains it even

where there is magnetisation. Now as we have just shown incidentally

$$V.\mathfrak{C}\mathfrak{H}_4 = V.\mathfrak{C}\mathfrak{B} - V.\mathfrak{C}2\pi\mathfrak{I}.$$

So Maxwell tacitly assumes that there is a force $V.\mathfrak{C}2\pi\mathfrak{I}$ acting from the element considered as a magnet, on the element considered as a conductor.

It is quite permissible to do this, in order to simplify the mathematical expressions, provided that, in reckoning up the forces acting on the element in virtue of its magnetism, we do not forget to include a force $-V.\mathfrak{C}2\pi\mathfrak{I}$ due to the current traversing the element. The addition of this force changes the x-component from

$$A\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dx} + B\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dy} + C\frac{d(\alpha_2+\alpha_3+\alpha_4)}{dz}$$

into

$$A\frac{d\alpha}{dx} + B\frac{d\beta}{dx} + C\frac{d\gamma}{dx},$$

as we have just shown.

If we reverse this imaginary force, we find the force on the element considered as a conductor is $V.\mathfrak{C}\mathfrak{H}$, and the x-component of the force on it considered as a magnet is

$$A\frac{d\alpha}{dx} + B\frac{d\alpha}{dy} + C\frac{d\alpha}{dz}.$$

The last is the same as we should obtain from first principles, if it were given that the magnetised element were placed in an undisturbed field $\mathfrak{H}$.

This pair of expressions is just as correct as the pair given by Maxwell, and leads of course to the same value of the total force on the element.

It might be contended that it would be more logical to suppose the imaginary force zero, but it is easy to show that the value of the imaginary force depends on the form of the element. The total force on an element, it is obvious, must depend merely on its volume, but the same is not true of the parts into which we arbitrarily divide the total force. In the case of a spherical element

$$V.\mathfrak{C}\mathfrak{B} = V.\mathfrak{C}\mathfrak{H}_4 + V.\mathfrak{C}\tfrac{8}{3}\pi\mathfrak{I},$$

so the imaginary force in Maxwell's expressions has the value $V.\mathfrak{C}\frac{8}{3}\pi\mathfrak{I}$.

To obtain Maxwell's expressions without any imaginary force coming in, we must use as an element a thin disc at right angles to the magnetisation, and to obtain the expressions suggested above, we must use a thin cylinder parallel to the magnetisation. With these elements it is easy to find the force on the element

considered as a conductor, but to complete the investigation it is necessary to find the $\mathfrak{H}_1$ due to an open circuit, and a number of difficulties are introduced.

By employing a cylinder, whose length is parallel to the current and is indefinitely great compared with its breadth, we get rid of all these difficulties. For since the form of the rest of the closed circuit only affects the values of $\mathfrak{H}_1$ in the negligibly small portions of the cylinder near the ends, it is reasonable to assume that the value of $\mathfrak{H}_1$ derived from any one of these closed circuits is the value due to the current in the cylinder alone, and this is the value we have used.

(3) *The problem of three moments.* By R. R. Webb, M.A.

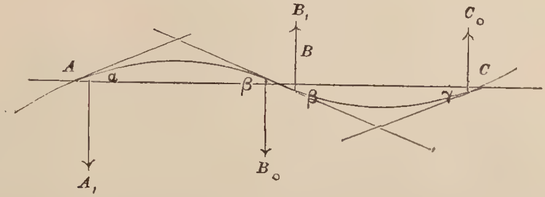
This problem originally due to Clapeyron (*Comptes Rendus*, xlv.) seems to be the basis of the practical determination of the bending moments in such structures as girder bridges. The history of the problem is concisely given in the Proceedings of the Royal Society (xviii. 1869—70) and there is no need of reproduction here.

The solution as given by Clapeyron (loc. cit.) proceeds on the comparatively simple hypotheses of *constant* flexural rigidity, *equal* spans, *uniform* loads, and the same is to be said of the extensive solution of the problem of the elastic beam resting on any number of supports as given by Bresse in his *Cours de Mécanique Appliquée* (Paris 1859). Rankine (Proceedings, Royal Society, xix. 1870—71) seems to have greatly remedied the defects of his predecessors, still his formulæ are encumbered with *double* integrals, and no serious attempt is made to meet the mathematical difficulty of discontinuous loading. Weyrauch (*Zeitschrift für Mathematik und Physik*, xviii, xix, 1873—4) gives general formulæ involving double integrals and meets the case of irregular loading with Fourier's analysis, a proceeding that virtually places his results to some extent outside the region of a practical computer.

The object of this note is to give an edition of the problem having the advantages of symmetry, while at the same time it meets the wants arising from (1) variable flexural rigidity, (2) unequal spans, (3) irregular loading. It will be convenient to firstly solve the problem when there is no load, and the rod is merely used as a stress transmitter. To fix ideas suppose the unstrained rod $O \ldots ABC \ldots$ to pass through smooth rings at $\ldots A, B, C \ldots$, let the shearing stress at any pier P be denoted by P_1 and P_0, fore and aft of the pier so that the pressure on this particular support will be $(P_1 \sim P_0)$, let M_A, M_B, M_C be the bending moments at A, B, C all measured positively in such a way as would in the

figure correspond bending the rod *over* each support. Then we have

$$\left.\begin{array}{r} A_1 \,.\, AB + M_A - M_B = 0 \\ -B_0 \,.\, AB + M_A - M_B = 0 \end{array}\right\},$$



so that $A_1 + B_0 = 0$ as is otherwise obvious, and the equation of upward deflection between A and B is,

$$-E\frac{d^2y}{dx^2} = M_A\frac{OB-x}{AB} + M_B\frac{x-OA}{AB} \text{ .... (i)},$$

where y is measured upwards, and x from O. Now multiply by $(x-OA)/E$, integrate from A to B and we have

$$+AB\tan\beta = M_A\int_{OA}^{OB}\frac{(OB-x)(x-OA)}{AB}\frac{dx}{E} + M_B\int_{OA}^{OB}\frac{(x-OA)^2}{AB}\frac{dx}{E}.$$

Whence we have

$$\tan\beta = M_A\int_{OA}^{OB}\frac{(OB-x)(x-OA)}{AB^2}\frac{dx}{E} + M_B\int_{OA}^{OB}\frac{(x-OA)^2}{AB^2}\frac{dx}{E} \text{..(ii)}.$$

Similarly, if between B and C the deflection be measured downwards, an exactly similar process leads to

$$\tan\beta = -M_C\int_{OB}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E} - M_B\int_{OB}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E} \text{..(iii)},$$

and on subtraction there results the equation

$$M_A\int_{OA}^{OB}\frac{(OB-x)(x-OA)}{AB^2}\frac{dx}{E} + M_C\int_{OB}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E}$$
$$+ M_B\left[\int_{OA}^{OB}\frac{(x-OA)^2}{AB^2}\frac{dx}{E} + \int_{OB}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E}\right] = 0 \text{...(iv)}.$$

We have now to pass on to the case in which a weight W is

placed at a point P of the beam between B and C. Adopting the same notation as before we now have

$$\left.\begin{aligned} B_1 \,.\, BC - W \,.\, PC - M_B + M_C = 0 \\ C_0 \,.\, BC - W \,.\, BP + M_B - M_C = 0 \end{aligned}\right\} \text{............ (v)},$$

and the equations of downward deflection are

$$\left.\begin{aligned} -E\frac{d^2y}{dx^2} &= -M_B + B_1(x - OB) \text{ from } B \text{ to } P \\ -E\frac{d^2y}{dx^2} &= -M_C + C_0(OC - x) \text{ from } P \text{ to } C \end{aligned}\right\} \text{......(vi)}.$$

On multiplying the former of these by $-(x - OB)/E$, and integrating from B to P, treating the second of these in like manner with $-(OC - x)/E$ and using limits that correspond to P and B, we get the equations

$$BP\left(\frac{dy}{dx}\right)_P - y_P = M_B\int_{OB}^{OP}(x - OB)\frac{dx}{E} - B_1\int_{OB}^{OP}(x - OB)^2\frac{dx}{E}$$

$$-PC\left(\frac{dy}{dx}\right)_P - y_P = M_C\int_{OP}^{OC}(OC - x)\frac{dx}{E} - C_0\int_{OP}^{OC}(OC - x)^2\frac{dx}{E},$$

leading on subtraction to

$$\begin{aligned} BC\left(\frac{dy}{dx}\right)_P = M_B\int_{OB}^{OP}(x - OB)\frac{dx}{E} - M_C\int_{OP}^{OC}(OC - x)\frac{dx}{E} \\ - B_1\int_{OB}^{OP}(x - OB)^2\frac{dx}{E} + C_0\int_{OP}^{OC}(OC - x)^2\frac{dx}{E} \text{......(vii)}. \end{aligned}$$

Referring back to the first of the two equations (vi) and integrating between the limits B and P we obtain

$$\left(\frac{dy}{dx}\right)_P - \tan\beta = M_B\int_{OB}^{OP}\frac{dx}{E} - B_1\int_{OB}^{OP}(x - OB)\frac{dx}{E},$$

and on eliminating $\left(\frac{dy}{dx}\right)_P$ between this and (vii) there results

$$\begin{aligned} \tan\beta = -M_B\int_{OB}^{OP}\frac{(OC - x)}{BC}\frac{dx}{E} - M_C\int_{OP}^{OC}\frac{(OC - x)}{BC}\frac{dx}{E} \\ + B_1\int_{OB}^{OP}\frac{(OC - x)(x - OB)}{BC}\frac{dx}{E} + C_0\int_{OP}^{OC}\frac{(OC - x)^2}{BC}\frac{dx}{E} \text{...(viii)}. \end{aligned}$$

Now result (ii) holds good here, though of course (iii) is modified into (viii), whence, substituting in (viii) the values of B_1,

C_0 and using (iii) as a guide to the form of the result of this substitution we get

$$\tan\beta = -M_B\int_{OB}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E} - M_C\int_{OB}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E}$$
$$+W\left[PC\int_{OB}^{OP}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E} + BP\int_{OP}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E}\right] \text{(ix)}.$$

Whence by means of (ii) and (ix) we now obtain

$$M_A\int_{OA}^{OB}\frac{(OB-x)(x-OA)}{AB^2}\frac{dx}{E} + M_C\int_{OB}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E}$$
$$+M_B\left[\int_{OA}^{OB}\frac{(x-OA)^2}{AB^2}\frac{dx}{E} + \int_{OB}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E}\right]$$
$$= W\left[PC\int_{OB}^{OP}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E} + BP\int_{OP}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{E}\right]. \text{(x)}.$$

The corresponding equation for the moments M_B, M_C, M_D will of course be

$$M_B\int_{OB}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E} + M_D\int_{OC}^{OD}\frac{(OD-x)(x-OC)}{CD^2}\frac{dx}{E}$$
$$+M_C\left[\int_{OC}^{OD}\frac{(OD-x)^2}{CD^2}\frac{dx}{E} + \int_{OB}^{OC}\frac{(OB-x)^2}{BC^2}\frac{dx}{E}\right]$$
$$= W\left[PC\int_{OB}^{OP}\frac{(OB-x)^2}{BC^2}\frac{dx}{E} + BP\int_{OP}^{OC}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{E}\right] \text{(xi)},$$

and in all other cases the equation is simply that given in (iv).

At present little further need be added other than the remark that the principle of superposition, or rather addition, enables us to build up from this case any other, no matter how complicated.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

January 31, 1887.

PROF. BABINGTON, VICE-PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

S. L. Loney, M.A., Sidney College.
R. A. Herman, M.A., Trinity College.
A. N. Whitehead, B.A., Trinity College.
E. G. Gallop, B.A., Trinity College.

The following communications were made to the Society:

(1) *On the Motion of a Ring in an Infinite Liquid.* By A. B. BASSET, M.A.

1. The general theory of the motion of a ring in an infinite liquid, when there is cyclic irrotational motion through its aperture, was I believe first given by Sir W. Thomson in the *Philosophical Magazine* for 1871, and his theory has been subsequently developed by Professor Lamb, in his *Treatise on the Motion of Fluids.* I propose in the present communication to apply these results to the discussion of certain special cases of the motion of a ring.

Let G be the centre of inertia of a plane curve S, OZ any fixed straight line lying in the plane of S, and let OG be perpendicular to OZ. In order not to be troubled with the products of velocities in the expression for the kinetic energy, I shall assume S to be symmetrical with respect to OG, but otherwise it may be of any form, provided there be no singular points capable of giving

rise to sharp edges; and the ring will be supposed to be generated by the revolution of S about OZ. Then O will be the centre of inertia of the ring, OZ its axis of unequal moment, which I shall call the *axis of the ring;* and I shall call the circle described by G the *circular axis of the ring.*

2. Let the ring be introduced into an infinite liquid which is at rest, and held fixed; let the circular aperture be closed up by means of a plane diaphragm, whose area is σ; and let cyclic irrotational motion be generated by applying to every point of this diaphragm a uniform impulsive pressure $\kappa\rho$, where ρ is the density of the liquid, and then let the diaphragm be removed.

The velocity potential of this cyclic motion will be

$$\phi = \kappa\Omega,$$

where Ω is a monocyclic function whose cyclic constant is unity, and κ is the circulation, round any closed circuit, which embraces the ring once only.

The resultant momentum of the cyclic motion will be parallel to the direction of the impulsive pressure in the diaphragm, and equal to ζ_0; and the energy to $K\kappa^2/2$, where

$$\zeta_0 = \kappa\rho\sigma - \kappa\rho\iint \Omega n dS,$$

$$K = \rho\iint \frac{d\Omega}{d\nu}\, d\sigma,$$

where n is the z-direction cosine of the normal to the ring drawn outwards, and dS an element of its surface.

If the ring be set in motion, the kinetic energy and momentum of the ring and liquid will be determined by the equations

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + K\kappa^2 \quad\ldots\ldots(1),$$

$$\left.\begin{array}{lll} \xi = Pu, & \eta = Bv, & \zeta = Rw + \zeta_0 \\ \lambda = A\omega_1, & \mu = A\omega_2, & \nu = C\omega_3 \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots(2).$$

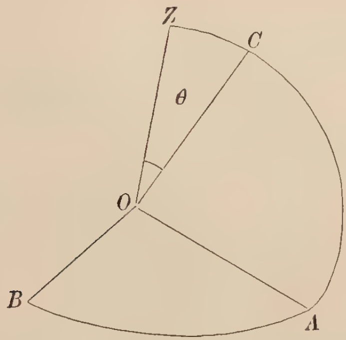
Since the liquid is incapable of producing a couple about the axis of the ring, $\omega_3 = \text{const.} = \omega$ throughout the motion.

Hence, if the ring be let go after the cyclic motion has been generated, it will remain at rest; for the only possible motion will be in the direction of its axis, and consequently

$$2T = Rw^2 + C\omega^2 + K\kappa^2 = \text{its initial value},$$

therefore $$w = 0.$$

3. Let the ring be set in motion by means of an impulsive couple G about any diameter OB of its circular axis.



The axis OC of the ring will evidently move in a fixed plane, which is perpendicular to the axis of the couple. Let θ be the inclination of OC to OZ at time t; u, w the velocities of O along OA and OC.

The principle of Conservation of Linear Momentum gives,

$$-\xi\sin\theta+\zeta\cos\theta=\zeta_0,$$

$$\xi\cos\theta+\zeta\sin\theta=0,$$

whence

$$\left.\begin{aligned} Pu &= -\zeta_0\sin\theta \\ Rw &= -\zeta_0(1-\cos\theta) \end{aligned}\right\}\dots\dots\dots\dots\dots\dots (3).$$

If $\dot{z}$, $\dot{x}$ be the velocities of O along and perpendicular to OZ, then

$$\dot{x}=u\cos\theta+w\sin\theta,$$

$$\dot{z}=w\cos\theta-u\sin\theta.$$

Therefore

$$\left.\begin{aligned} \dot{x} &= \zeta_0\left(\frac{1}{R}-\frac{1}{P}\right)\sin\theta\cos\theta-\frac{\zeta_0}{R}\sin\theta \\ \dot{z} &= \frac{\zeta_0}{P}+\zeta_0\left(\frac{1}{R}-\frac{1}{P}\right)\cos^2\theta-\frac{\zeta_0}{R}\cos\theta \end{aligned}\right\}\dots\dots\dots\dots (4).$$

Also

$$2T=Pu^2+Rw^2+A\dot{\theta}^2+K\kappa^2=\text{const.}$$

Substituting the values of u and w from (3) we obtain,

$$A\dot{\theta}^2 = A\omega^2 - \zeta_0^2\left(\frac{1}{P}+\frac{1}{R}\right) + \frac{2\zeta_0^2}{R}\cos\theta + \zeta_0^2\left(\frac{1}{P}-\frac{1}{R}\right)\cos^2\theta \dots (5)$$

$$= f(\theta) \text{ say,}$$

where ω is the initial value of $\dot{\theta}$.

The character of the motion depends upon the roots of the equation $f(\theta) = 0$, which we shall now consider.

The roots are

$$\cos\theta = \frac{-\dfrac{\zeta_0}{R} \pm \sqrt{\left\{\dfrac{\zeta_0^2}{P^2} - A\omega^2\left(\dfrac{1}{P}-\dfrac{1}{R}\right)\right\}}}{\zeta_0\left(\dfrac{1}{P}-\dfrac{1}{R}\right)}.$$

Case I. Let $R > P$.

In order that the roots may be real, we must have

$$\omega < \zeta_0 \sqrt{\frac{R}{AP(R-P)}}.$$

If this condition be satisfied, one root will be positive and < 1, and the other will be negative and less than -1. Hence $\dot{\theta}$ will vanish when θ has some value β lying between 0 and $\pi/2$, and the ring will oscillate between the angles β and $-\beta$.

But if
$$\omega > \zeta_0 \sqrt{\frac{R}{AP(R-P)}}$$
both roots will be imaginary, and $\dot{\theta}$ will never vanish. Hence the ring will make a complete revolution.

Case II. Let $P > R$.

In this case both roots are real, and one of them is positive and < 1 provided ω be sufficiently small; but if ω be sufficiently large both roots will be negative and < -1. In order that one root should not be < -1, it is necessary that

$$\omega < \frac{2\zeta_0}{\sqrt{AR}}.$$

If this condition be satisfied, the ring will oscillate between the angles β and $-\beta$, where β lies between 0 and π; but if

$$\omega > \frac{2\zeta_0}{\sqrt{AR}},$$

the ring will make a complete revolution.

In order to find the period of oscillation or revolution, as the case may be, we must express θ in terms of t.

Case I. $R > P$.

(i) Let the roots be real and equal to p and $-q$, where

$$q > 1 > p > 0.$$

Equation (5) may be written

$$\dot{\theta}^2 = M^2 (\cos\theta - p)(\cos\theta + q),$$

where

$$M^2 = \frac{\zeta_0^2}{APR}(R - P).$$

Let

$$\cos\theta = \frac{1 - D\cos^2\phi}{1 + D\cos^2\phi},$$

where

$$D = \frac{1-p}{1+p}.$$

Then

$$d\theta = \frac{2\sqrt{D}\sin\phi\, d\phi}{1 + D\cos^2\phi},$$

$$(\cos\theta - p)(\cos\theta + q) = \frac{(1-p)(1+q)}{(1 + D\cos^2\phi)^2}(1 - k^2\sin^2\phi),$$

where

$$k^2 = \frac{(q-1)(1-p)}{2(p+q)}.$$

Therefore
$$Mdt = \frac{2}{\sqrt{(1+p)(1+q)}} \frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}},$$

therefore

$$\phi = amIt,$$

where

$$I = \tfrac{1}{2}M\sqrt{(1+p)(1+q)}.$$

Therefore

$$\cos\theta = \frac{1 + p - (1-p)\,\text{cn}^2 It}{1 + p + (1-p)\,\text{cn}^2 It},$$

and the period of a complete oscillation is $4K/I$.

(ii) Let the roots be imaginary and equal to $p \pm \iota q$.

Then

$$\dot{\theta}^2 = M^2\{(\cos\theta - p)^2 + q^2\}.$$

Let

$$\cos\theta = \frac{1 - D + (1 + D)\cos\phi}{1 + D + (1 - D)\cos\phi}.$$

Then $$d\theta = \frac{2\sqrt{D}\,d\phi}{1+D+(1-D)\cos\phi},$$

and

$$\{(\cos\theta-p)^2+q^2\}\{1+D+(1-D)\cos\phi\} = \{1-D-p(1+D)\}^2+q^2(1+D)^2$$
$$+2\cos\phi\,[(1-p)^2+q^2-D^2\{(1+p)^2+q^2\}]+[\{1+D-p(1-D)\}^2$$
$$+q^2(1-D)^2]\cos^2\phi.$$

Hence, if $$D^2 = \frac{(1-p)^2+q^2}{(1+p)^2+q^2},$$

the coefficient of $\cos\phi$ will vanish; substituting this value of D, we obtain

$$\frac{d\theta}{\sqrt{\{(\cos\theta-p)^2+q^2\}}} = \frac{1}{\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{4}}}\,\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},$$

where $$k^2 = \frac{1}{2}\left[1+\frac{1-p^2-q^2}{\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{2}}}\right].$$

Hence $$\phi = amI't,$$

where $$I' = M\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{4}};$$

and we finally obtain

$$\tan^2\frac{\theta}{2} = \sqrt{\frac{(1-p)^2+q^2}{(1+p)^2+q^2}}\cdot\frac{1-\text{cn}\,I't}{1+\text{cn}\,I't},$$

and the time of a complete revolution is $4K/I'$.

Case II. $P > R$.

In this case both roots are real, and one root is always negative and numerically greater than unity.

(i) Let the roots be p and $-q$, where $q>1>p>0$. The transformation is the same as in Case I. sub-case (i).

(ii) Let the roots be $-p$ and $-q$, where $q>1>p>0$.

Then $$\dot\theta^2 = M^2(\cos\theta+p)(\cos\theta+q),$$

where $$M^2 = \frac{\zeta_0^2}{APR}(P-R).$$

In this case we employ the same transformation, but must put

$$D = \frac{1+p}{1-p},$$

$$k^2 = \frac{D(q-1)}{1+q+D(q-1)} = \frac{(q-1)(1+p)}{2(q-p)}.$$

(iii) Let the roots be $-p$ and $-q$, where $q>p>1$.

We must put
$$\cos\theta=\frac{1-D\sin^2\phi}{1+D\sin^2\phi},$$
where
$$D=-\frac{p-1}{p+1},$$
$$k^2=\frac{(p-1)(q-1)}{(p+1)(q+1)}.$$

In order to obtain the path described by the centre of inertia O of the ring, we must substitute the value of θ in terms of t in (4), and integrate the result.

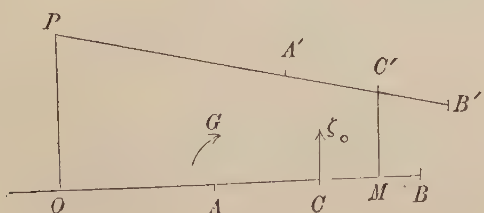
We can however ascertain the character of the motion of O without integrating (4). For differentiating (5) we obtain
$$A\ddot{\theta}=-\frac{\zeta_0^2}{R}\sin\theta-\zeta_0^2\left(\frac{1}{P}-\frac{1}{R}\right)\sin\theta\cos\theta.$$

Therefore
$$\dot{x}=\frac{A\ddot{\theta}}{\zeta_0},$$
and
$$x=\frac{A}{\zeta_0}(\dot{\theta}-\omega).$$

Also the value of $\dot{z}$ may be written
$$\dot{z}=\frac{\zeta_0}{P}-\left[\zeta_0\left(\frac{1}{P}-\frac{1}{R}\right)\cos^2\theta+\frac{\zeta_0}{R}\cos\theta\right].$$

The term in square brackets has its greatest value when $\theta=0$, in which case $\dot{z}=0$; hence $\dot{z}$ can never become negative. The motion of O is such that O moves along the initial direction of the axis of the ring with a uniform velocity, superimposed upon which is a variable periodic velocity; and at the same time vibrates perpendicularly to this line.

[4. The equations of motion in the last article may be obtained in a different manner by means of Sir W. Thomson's Theory of the Impulse.



Let AB be the initial position of the projection of the ring on the paper, which is supposed to be perpendicular to its plane.

The impulsive forces which must be applied to the ring and barrier in order to produce the initial motion, consist of a linear impulse ζ_0, and a couple $G = A\omega$, about the diameter which is perpendicular to AB. Hence, if $OC = A\omega/\zeta_0$, the impulse of the whole motion consists of a linear impulse ζ_0 along OP.

Since there are no impressed forces, it follows that if $A'B'$ be the position of the ring at any subsequent time, the motion must be such that it could be instantaneously produced by applying to the ring and barrier an impulse ζ_0 at P along OP. This impulse may be supposed to be applied by means of proper mechanism, connecting the ring and barrier with P.

If θ be the angle which the axis of the ring makes with OP, the conditions that the force constituent of the impulse should be equal to ζ_0 lead to equations (3). The condition that the couple constituent should vanish gives

$$A\dot{\theta} - (Rw + \zeta_0)\, PC' = 0,$$

or

$$A\dot{\theta} = \zeta_0\, PC' \cos\theta = \zeta_0 x + A\omega.$$

Differentiating with respect to t, substituting the value of $\dot{x}$ in terms of θ from (4), and then integrating, we shall obtain (5).

Since the momentum due to the circulation alone is always perpendicular to the plane of the ring, it follows that if a ring initially at rest be set in motion by means of a couple about a diameter, the direction of this momentum will be changed; and the opposition which the liquid exerts against this action on the part of the ring will produce a couple tending to oppose the rotation of the ring. Also, since the impressed couple can produce no effect on the linear momentum of the system, it follows that the effect of changing the direction of the momentum due to the circulation, will be to cause the ring to move with a velocity of translation, which gives rise to a linear component of momentum of the whole *system*, such that the resultant of the latter and ζ_0 (whose direction has been changed) must be a momentum equal to ζ_0, and whose direction coincides with the original direction of ζ_0.]

5. We shall now investigate the stability of the motion of a ring, which is moving parallel to its axis in the direction of the cyclic motion.

When the motion is steady

$$\zeta = Rw + \zeta_0 = \text{const.} = \gamma,$$
$$\nu = C\omega_3 = \text{const.} = C\Omega,$$
$$\xi = \eta = \lambda = \mu = 0.$$

In order to obtain the disturbed motion, we must have recourse to Kirchhoff's equations of motion*; we shall also suppose that the co-ordinate axes are fixed in the ring.

Putting for shortness

$$Z = \gamma + \frac{P(\zeta_0 - \gamma)}{R},$$

the equations of disturbed motion are,

$$P\dot{u} - P\Omega v + \gamma\omega_2 = 0,$$
$$P\dot{v} - \gamma\omega_1 + P\Omega u = 0,$$
$$A\dot{\omega}_1 + Zv + (C-A)\,\Omega\omega_2 = 0,$$
$$A\dot{\omega}_2 - Zu - (C-A)\,\Omega\omega_1 = 0.$$

Whence, if p be the period of oscillation,

$$\begin{vmatrix} Pp & -P\Omega & 0 & \gamma \\ P\Omega & Pp & -\gamma & 0 \\ 0 & Z & Ap & (C-A)\,\Omega \\ -Z & 0 & -(C-A)\,\Omega & Ap \end{vmatrix} = 0,$$

or

$$A^2P^2p^4 + P[2ZA\gamma + \{(C-A)^2 + A^2\}P\Omega^2]\,p^2 + \{P(C-A)\,\Omega^2 + Z\gamma\}^2 = 0.$$

The two values of p^2 given by this equation are both real and negative. Hence the values of p are imaginary, and therefore the motion is stable.

If $\Omega = 0$, the roots are

$$p = \pm\iota\sqrt{\frac{Z\gamma}{AP}},$$

and are imaginary provided $R\gamma + P\zeta_0 > P\gamma$. Hence, if $R > P$ the motion, when there is no rotation, is always stable; but if $P > R$ it may be unstable.

[If a prolate solid be moving parallel to its axis and there is no circulation, the motion will be unstable unless the angular velocity about its axis is sufficiently great; and the preceding article shows that when there is circulation but no angular velocity,

* Lamb, *Motion of Fluids*, p. 126.

the motion will be stable provided there is sufficient circulation; but when there is both circulation and rotation about the axis, the motion is always stable. A general explanation of the reason of this has already been given in Art. 4.]

6. Another kind of steady motion may be obtained by setting the ring in motion by means of a couple G about a diameter of its circular axis, and at the same time applying an impulse ζ_0 in the opposite direction to that of the cyclic motion.

The effect of the latter impulse is to destroy the linear momentum of the system, hence

$$\xi = 0, \quad \zeta = 0.$$

Therefore $$u = 0, \quad w = -\frac{\zeta_0}{R}.$$

Kirchhoff's 5th equation gives

$$\mu = \text{const.} = G = A\dot{\theta}.$$

The motion of the ring is such that its centre of inertia O, describes a circle about a fixed axis parallel to the axis of the couple, through which the plane of the ring always passes. If r be the distance of O from this axis,

$$\frac{\zeta_0}{R} = -w = r\dot{\theta} = \frac{Gr}{A};$$

$$\therefore \quad r = \frac{A\zeta_0}{RG}.$$

In order to determine the stability, we must put in the general equations of motion,

$$\xi = Pu, \qquad \eta = Pv, \qquad \zeta = Rw,$$

$$\lambda = A\omega_1, \qquad \mu = G + A\omega_2, \qquad \nu = 0,$$

$$w = -\frac{\zeta_0}{R} + w, \qquad \omega_2 = -\frac{G}{A} + \omega_2,$$

where the quantities u, v, &c., on the right-hand sides of these equations, are small quantities in the beginning of the disturbed motion. Also, if the axes are fixed in the ring,

$$\theta_1 = \omega_1, \qquad \theta_2 = \frac{G}{A} + \omega_2, \qquad \theta_3 = 0,$$

and the equations of disturbed motion are

$$P\dot{u} + \frac{RG}{A} w = 0,$$

$$P\dot{v} = 0,$$

$$R\dot{w} - \frac{PG}{A} u = 0,$$

$$A\dot{\omega}_1 + \frac{P\zeta_0}{R} v = 0,$$

$$A\dot{\omega}_2 - \frac{P\zeta_0}{R} u = 0.$$

From the first and third equations we obtain

$$w = w' \sin\left(\frac{Gt}{A} + \alpha\right),$$

$$u = \frac{Rw'}{P} \cos\left(\frac{Gt}{A} + \alpha\right).$$

The fifth equation gives

$$\omega_2 = \frac{\zeta_0 w'}{G} \sin\left(\frac{Gt}{A} + \alpha\right) + \text{const.}$$

The second and fourth give

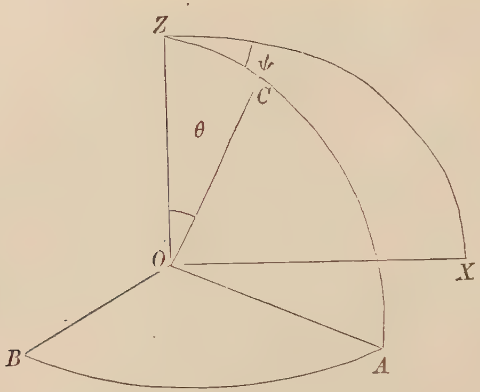
$$v = \text{const.},$$

$$\omega_1 = -\frac{P\zeta_0}{AR} vt + \text{const.}$$

These equations show that the motion is stable for all displacements which do not tend to remove the centre of inertia from the plane of its motion; but the motion is unstable for all displacements which tend to produce this effect. If the disturbance is such that $v = 0$, the disturbed motion will still be stable, but the axis of rotation will be shifted through a certain angle.

7. A third kind of steady motion, which is helicoidal, is obtained by first communicating to the ring an arbitrary angular velocity Ω about its axis; secondly by applying an impulsive couple G about an axis inclined at an arbitrary angle a to the axis of the ring; and thirdly by applying a determinate impulse in the plane of the axes of the ring and couple.

In order that steady motion may be possible, it is necessary that v and therefore η should be zero throughout the motion. This condition may be secured by means of an impulsive force whose components are $X = -\zeta_0 \sin\alpha$, $Z = F$.



The equations of momentum are

$$(\xi \cos\theta + \zeta \sin\theta)\cos\psi - \eta \sin\psi = 0,$$
$$(\xi \cos\theta + \zeta \sin\theta)\sin\psi + \eta \cos\psi = 0,$$
$$-\xi \sin\theta + \zeta \cos\theta = F + \zeta_0 \cos\alpha\,;$$

whence

$$\left.\begin{aligned}\xi &= -(F + \zeta_0 \cos\alpha)\sin\theta\\ \eta &= 0\\ \zeta &= (F + \zeta_0\cos\alpha)\cos\theta\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Since the components of momentum parallel to the axes of X and Y (which are fixed in direction, but not in position because O is in motion) are zero throughout the motion, the angular momentum about OZ is constant, whence

$$-A\omega_1 \sin\theta + C\Omega\cos\theta = G + C\Omega\cos\alpha \;\ldots\ldots\ldots\ldots(7).$$

The equation of energy gives

$$Pu^2 + Rw^2 + A(\omega_1^2 + \dot\theta^2) = \text{const.},$$

therefore

$$\frac{(F+\zeta_0\cos\alpha)^2\sin^2\theta}{P} + \frac{\{(F+\zeta_0\cos\alpha)\cos\theta - \zeta_0\}^2}{R} + \frac{\{G + C\Omega(\cos\alpha - \cos\theta)\}^2}{A\sin^2\theta}$$
$$+ A\dot\theta^2 = \text{const.} = \text{its initial value}\ldots\ldots\ldots\ldots\ldots(8).$$

This equation determines the inclination θ of the axis.

So far our equations have been perfectly general, we shall now introduce the conditions of steady motion. These are

$$\theta = \alpha, \qquad \dot\psi = \mu, \qquad \ddot\theta = \dot\theta = 0 \;\ldots\ldots\ldots\ldots\ldots(9),$$

whence (7) becomes

$$A\mu\sin^2\alpha = G \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Differentiating (8) with respect to t, and using (9) and (10), we obtain

$$A\mu^2 \cos\alpha - C\Omega\mu + \left(\frac{1}{R} - \frac{1}{P}\right) Z^2 \cos\alpha - \frac{Z\zeta_0}{R} = 0 \;\ldots\; (11),$$

where $$Z = F + \zeta_0 \cos\alpha.$$

In order that steady motion may be possible, we must have

$$C^2\Omega^2 > 4ZA\cos\alpha \left[\left(\frac{1}{R} - \frac{1}{P}\right) Z\cos\alpha - \frac{\zeta_0}{R}\right] \;\ldots\ldots\; (12).$$

Hence, if $R > P$ steady motion will always be possible, but if $P > R$, steady motion will be impossible unless the condition (12) is satisfied.

If x, y, z be the co-ordinates of O, we have

$$\dot{x} = (u\cos\theta + w\sin\theta)\cos\psi = \left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\zeta_0}{R}\right\}\sin\alpha\cos\mu t,$$

$$\dot{y} = (u\cos\theta + w\sin\theta)\sin\psi = \left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\zeta_0}{R}\right\}\sin\alpha\sin\mu t,$$

$$\dot{z} = w\cos\theta - u\sin\theta = Z\left(\frac{\sin^2\alpha}{P} + \frac{\cos^2\alpha}{R}\right) - \frac{\zeta_0\cos\alpha}{R};$$

whence the centre of inertia describes the helix

$$x = \frac{1}{\mu}\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\zeta_0}{R}\right\}\sin\alpha\sin\mu t,$$

$$y = -\frac{1}{\mu}\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\zeta_0}{R}\right\}\sin\alpha\cos\mu t,$$

$$z = \left\{Z\left(\frac{\sin^2\alpha}{P} + \frac{\cos^2\alpha}{R}\right) - \frac{\zeta_0\cos\alpha}{R}\right\} t.$$

This last result may be at once obtained from the fact that the impulse of the motion must consist of wrench about a fixed axis.

To examine the stability differentiate (8) with respect to t, and we obtain

$$A\ddot{\theta} + f(\theta) = 0.$$

Hence the motion will be stable or unstable according as $f'(\alpha)$ is positive or negative.

Now

$$f(\theta) = \frac{Z^2}{2}\left(\frac{1}{P} - \frac{1}{R}\right)\sin^2\theta + \frac{Z\zeta_0\sin\theta}{R} + \frac{C\Omega}{A\sin\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\}$$

$$- \frac{\cos\theta}{A\sin^3\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\}^2;$$

therefore

$$p^2 = f'(\alpha) = A\mu^2 (1 + 2\cos^2\alpha)$$
$$- 3\,C\Omega\mu\cos\alpha + \frac{C^2\Omega^2}{A} - Z^2\left(\frac{1}{R} - \frac{1}{P}\right)\cos^2\alpha + \frac{Z\zeta_0}{R}\cos\alpha.$$

Eliminating Ω by means of (11) we obtain

$$A^2p^2\mu^2 = A^2\mu^4 + A\mu^2 Z\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 - 3\cos^2\alpha) + \frac{2\zeta_0}{R}\cos\alpha\right\}$$
$$+ Z^2\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\zeta_0}{R}\right\}^2.$$

The right-hand side is positive, hence both positions of steady motion given by (11) are stable.

(2) *On the Form and Position of the Horopter.* By J. Larmor, M.A., St John's College.

1. When the two eyes are kept converged upon a fixed point, the images of another point will usually fall upon non-corresponding points of their retinas, and it will therefore be seen double. But there is a system of points forming a curved line in the field of view which are such that the images correspond, and they are therefore seen single. The locus of points possessing this property was called the horopter, first by Aquilonius. It is of importance in the theory of stereoscopic vision as defining the neighbourhood in which the images formed by the two eyes are perfectly fused together; and accordingly its properties have been investigated by Helmholtz, Hering, and other physiologists.

The final investigation of Helmholtz was published in 1867, and presents the theory under an analytical form. Geometrically it is a case of the theory of linear congruences of the first order, and forms a good example of the Geometry of Rays which has been explicitly introduced and applied chiefly by Plücker and his successors, since Helmholtz's papers were published.

It may be of advantage to give a brief account of the way in which the general results flow directly from the geometrical relations without recourse to symbolical reasoning. The direction of the horopter curve where it passes through the point of vision will then be investigated, as that would appear to be the most important matter for practical purposes on account of the complexity of the complete equations of the curve in the general case.

2. The eye being an optical instrument symmetrical round an axis, the aspect of external objects which is presented to its external

nodal point is projected unchanged from its internal nodal point upon the retina.

The nodal points of the eye are usually taken as coincident. This can always be done in an optical instrument, without sensible error, if their distance apart is a small fraction of the distances of the points in the field of vision; and in the case of the eye they almost coincide. But the theory which follows applies equally well without this simplification, as is evident from the remark at the beginning of this section.

Corresponding points on the retinas are those which would coincide when they are superposed without perversion, *i.e.* right corresponds to right, and left to left.

3. In the first place, it is to be remarked that curved lines can be constructed to any extent which are seen by single vision with both eyes focussed on some given point. For draw any curve on one retina and the corresponding curve on the other; join these to the internal nodal points by cones; transfer the vertices of these cones to the external nodal points without introducing any rotation; the intersection of the two cones will then be a curve possessing the property in question.

For example, suppose the curves on the retinas are conics; the cones will be quadric cones; their curve of intersection may include a conic, in which case the remaining part of it is another conic. If therefore a conic curve in space be such that it is seen singly with both eyes, there is another conic in space which is seen by both eyes in coincidence with it, and undistinguishable from it so long as no accommodation of the eyes is allowed.

4. But the simplest group of figures of this kind is that of the straight lines in space which are seen singly. They are constructed as before: draw two corresponding lines on the retinas, and join them by planes to the internal nodal points; the parallel planes drawn through the external nodal points intersect on a line of the group.

Now all lines in space may be viewed as the intersections of planes, one passing through each of these nodal points; but the intersecting planes here considered are allied to each other by two lineo-linear (because projective) relations derived from the correspondence of their intersections with two given planes (the retinas).

Their lines of intersection therefore form a congruence of the first order, in the general sense that through any point in space one line of the congruence can be drawn.

This property also appears more directly in the following manner. Consider any point in space; mark its corresponding points on the two retinas; these will not usually themselves

correspond, so mark the point on the other retina that corresponds to each; these points determine two lines on the retinas which correspond to each other; the line in space constructed from them passes through the point considered. Thus through any point in space passes usually one, and only one, line of the system.

Now it was shown by Sir W. R. Hamilton*, that the well-known propositions of Malus respecting the shortest distances of a normal to a surface from the consecutive normals, and the locus of their points of intersection, can all be extended to the general case of a system of rays which satisfy two conditions. One of these propositions is that each ray of the system is intersected at two points on its length by a consecutive ray, and therefore that each ray of the system is a bitangent to a focal surface which is the locus of these points, and is the analogue of the surface of centres in the simpler case of the normals.

In the system under consideration, one, and only one, ray can usually be drawn through any point in space; therefore a point of intersection of two consecutive rays is a singular point, and must be the point of intersection of an infinite number of rays, forming a cone. The focal surface, which is the locus of such points, must therefore degenerate into a curve. Further, through any point in space can be drawn one line which meets this curve twice; therefore all conical projections of the curve possess only one double point; therefore the curve is a twisted cubic, and is the partial intersection of two quadric surfaces, but it may degenerate into two lines†.

It may be here remarked that the congruence of the first order of Plücker is a more special form, corresponding to two linear relations between the six co-ordinates of the ray: for it the focal surface degenerates into two straight lines, each of which is met by all rays of the congruence.

5. The results just proved, when modified by projection, give geometrical theorems of interest as being the extensions to space of three dimensions of well-known plane theories. Thus, if two homologous systems of planes pass through two given points, the lines of intersection of corresponding planes are the chords of a twisted cubic curve: if two homologous systems of rays pass through two fixed points, the points of intersection of those corresponding rays which intersect lie on a twisted cubic curve. Cases in which the cubic breaks up into a line and a plane conic are examined in detail by Helmholtz.

* [Previously by Monge in 1781: see Prof. Cayley in *Proc. Lond. Math. Soc.* XIV. p. 139.]

† Salmon's *Solid Geometry*, Appendix III.: Salmon, *loc. cit.* § 364.

6. The curves and lines hitherto considered are seen singly because their images occupy corresponding lines on the retinas; but it is not necessary for this that the images of a definite point on one of them should occupy corresponding points. This would be a more difficult condition to fulfil, and only holds for the points of intersection of lines of the group, for then the image points are determined as lying on each of two lines. It is in fact clear that through such a point a singly infinite series of lines of the group can be drawn, forming a cone: for through its images on the retinas a singly infinite number of corresponding pairs of lines can be drawn, and each pair determines one of the group*. Thus the locus of points seen singly is the twisted cubic curve which is the nodal curve of the congruence.

It is worth while to point out that the pairs of points on one of the lines of the congruence whose images occupy corresponding positions on the retinas form a geometrical involution; that the line meets the horopter curve in its double points; and that its foci are therefore equidistant from these points.

7. If a third condition is given between the parameters of the line, the locus of the line becomes a ruled surface; if this new condition is linear, *e.g.* if the images of the line on the retinas correspond under normal conditions to a horizontal line or to a vertical line in space, the surface is a ruled hyperboloid.

These two surfaces are the horizontal and vertical line horopters of Helmholtz, and the cubic curve is the point horopter.

8. Inasmuch as the field of simultaneous vision of the eye is necessarily small, the most important part of the point horopter is that in the neighbourhood of the point on which the eyes are fixed. And it may be observed also, that the more oblique portions of the field of view do not practically come under the conditions of the geometrical problem, for the image on the retina cannot be considered as plane, except in its central portions.

The horopter may therefore be identified with the tangent to it at that point for most practical purposes. It seems therefore of importance to obtain the direction of this tangent line, especially as the results come out to be comparatively simple. At all points in the field of view in the neighbourhood of this line, the binocular vision will preserve completely the single character.

The result obtained will apply immediately to all the simpler cases in which the horopter curve breaks up into a line and a conic, the only ones that have been completely discussed†. For the more general case the angle of rotation ϕ of this section is given

* Helmholtz, *Wissen. Abhandl.* II. p. 488; *Physiological Optics*, § 31.

† Helmholtz, *Physiological Optics*, § 31; Hermann, *Physiologie*, 7 ed. pp. 419–25.

only approximately by the law of Listing; which is an additional reason for the sufficiency of the result here obtained.

Let then a_1, a_2 be the distances of the point of vision from the external nodal points of the two eyes, and 2γ the angle between the axes of vision of the two eyes. Let the radius of the second retina corresponding to the radius of the first which is in the plane of the axes of vision make an angle ϕ with that plane, owing to the action of the converging muscles.

Consider two right cones of equal small angle α round the axes of vision of the two eyes as axes, and suppose the corresponding generating lines on them to be marked in such way as to identify them. These cones will intersect in a curve, and at two opposite points on this curve corresponding generators will meet one another, but at no other points. These two points are situated on the point horopter, and determine its direction in the neighbourhood of the point of vision.

Taking for axis of x the bisector of the internal angle between the axes of vision drawn outwards, for axis of y the bisector of the external angle between the same lines drawn towards the first of them, and for axis of z the normal to the plane of the same lines, the co-ordinates of this point P of the horopter are easily determined. For, draw PM perpendicular to the plane of the axes of vision, and MN_1, MN_2 perpendicular to the axes of the two eyes. If θ denote the azimuth of P round the axis of the first cone, measured towards z from the positive direction of the axis of y, then $\theta+\phi$ will be its azimuth round the axis of the second cone. We therefore have the equations

$$PM \equiv z = \alpha a_1 \sin\theta \quad = \alpha a_2 \sin(\theta+\phi) \quad \ldots\ldots\ldots(1)$$

$$MN_1 \equiv x\sin\gamma + y\cos\gamma = \alpha a_1 \cos\theta \quad \ldots\ldots\ldots(2)$$

$$MN_2 \equiv -x\sin\gamma + y\cos\gamma = \alpha a_2 \cos(\theta+\phi) \quad \ldots\ldots\ldots(3)$$

Care has been taken to introduce only distances that are multiplied by the small quantity α, so that it is admissible to write a_1 and a_2 for the distances of P from the nodal points.

Thus $$\frac{x\sin\gamma + y\cos\gamma}{a_1\cot\theta} = \frac{-x\sin\gamma + y\cos\gamma}{a_2(\cot\theta\cos\phi - \sin\phi)} = \frac{z}{a_1} \quad \ldots\ldots(4)$$

where, by (1) $$\cot\theta = \frac{-a_2\cos\phi + a_1}{a_2\sin\phi}.$$

Substituting this value in (4),

$$\frac{x\sin\gamma + y\cos\gamma}{-a_1a_2\cos\phi + a_1^2} = \frac{-x\sin\gamma + y\cos\gamma}{a_1a_2\cos\phi - a_2^2} = \frac{z}{a_1a_2\sin\phi};$$

therefore

$$\frac{x \sin\gamma}{a_1^2 + a_2^2 - 2a_1a_2\cos\phi} = \frac{y\cos\gamma}{a_1^2 - a_2^2} = \frac{z}{2a_1a_2\sin\phi} \quad \text{.......(5)}$$

These values of x, y, z are proportional to the direction cosines of the tangent to the horopter curve.

They shew that when a_1 exceeds a_2, and ϕ is measured round in such direction that it is less than two right angles, the horopter lies in the quadrant of the field of vision which ranges from a_1 in the positive direction, and that it slopes away from the eyes in that quadrant.

Further approximation in equations (1), (2), (3) would lead easily to the determination of its curvature, but the results are too complicated to be of much use.

9. When the point of vision is in the medial plane, $a_1 = a_2$, and the direction of the horopter is in the medial plane, inclined to the plane of vision at an angle whose tangent is

$$\sin\gamma \cot\frac{1}{2}\phi;$$

and it slopes away from the eyes in the upward direction.

When $\phi = 0$, the direction is in the plane of vision, and makes an angle with the axis of x whose tangent is

$$\frac{a_1 + a_2}{a_1 - a_2}\tan\gamma.$$

These are in agreement with known results: in the first case the arc in question is part of a line, in the second it is part of Müller's circle.

(3) *On the finer structure of the walls of the endosperm cells of Tamus communis.* By Walter Gardiner, M.A.

It would appear from the author's more recent researches that the perforation of the walls of the endosperm cells in the plant referred to, is established after the formation of the wall, and in a similar manner to that which occurs in sieve-tubes during the formation of the sieve-plate. The author further hopes to shew that this is a special instance of a phenomenon.

February 14, 1887.

MR TROTTER, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society.

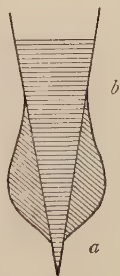
(1) *On the Influence of Capillary Action in some Chemical Decompositions.* By G. D. LIVEING, M.A.

There are numerous cases in which chemical action is promoted by means which are believed to be purely mechanical, but where the precise nature of the mechanical action is quite undefined, or only vaguely indicated by some analogy which is not clearly made out. Such cases are the rapid decomposition of hydrogen peroxide and hydrogen persulphide in the presence of many substances in fine powder, and of potassium chlorate in the presence of manganese black oxide, and many actions called katalytic. The action in some of these cases may, I think and shall try to shew, be referred to the same causes as those which produce the common effects of capillarity.

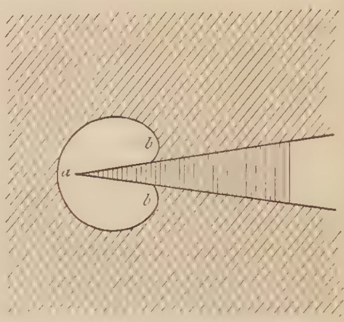
There is no doubt that solids condense on their surfaces films of air and other gases in which they may be immersed; and if we accept Laplace's theory of capillarity it seems impossible to escape the conclusion that the molecular attractions which produce surface tension must act upon, and condense, the molecules of gases in contact with the surfaces. Indeed the surface tension at the common boundary of two fluids depends on the nature of both fluids, even when one of them is a gas and can hardly be said to have any surface tension of its own. The condensation upon hygroscopic bodies of aqueous vapour from the atmosphere at temperatures above the dew point is recognised to be the result

of capillary action. The same cause will also account for the condensation of other vapours on solids and liquids when the space containing them is not saturated with those vapours; and there is no essential difference between condensible vapours and the so-called incondensible gases which could affect the result except in degree. We should expect all vapours and gases to be condensed on the surfaces of solids and liquids more or less.

No one can have observed the way in which bubbles of air form upon a piece of metal immersed in hot water, and compared it with the way in which globules of oil form on the surface of a glass plate which has been wetted with oil and then immersed in water, without being struck by the similarity of the appearances. The film of air in the one case, and of oil in the other, is rolled up into globules which adhere to the plate by a portion of their surface. The explanation given of the one will serve equally well for the other. In the case of the oiled glass the surface tension for water and glass is less than the difference of the tensions for oil and glass and for oil and water. So too for the metal plate and its film of air, the air is rolled up into bubbles because the surface tension between water and the metal is less than the difference of the surface tensions for air and metal and for air and water. The bubbles of air, and the globules of oil, remain adhering to the surface of the solid only when it has a tolerably large radius of curvature, but escape with great readiness from points, i.e. from surfaces of small radius of curvature. This agrees with what is observed in the case of drops. A large drop will hang from a large surface, such as the blunt end of a rod, if thoroughly wetted by it, but you cannot get a drop of oil to hang from the point of a needle or a drop of water from a fine pointed glass rod. The drop if it holds on to the solid always takes a form something like that shewn in the adjoining figure: the point sticks out through the drop. And if you want small drops to fall readily you drop them from a rod drawn out to a fine end. However to form drops like that in the figure the surface tension between the solid and adhering liquid must be less than that between air and the solid. If the film of air on a pointed solid is rolled up to the point by the capillary action of the liquid wetting the solid it will have a small surface to adhere to and will be easily disengaged just as small drops fall from a pointed solid. But there is another reason which seems to me to conduce to this result. When the surface tension between the solid and liquid is small the surface of the drop is concave to the air at points a and b, the radius of curvature at a is less than that at b in the principal section,



and the effect is to drive the drop up the solid until its increased size and weight counteract this tendency. If however the bubble be one of air on the point of a solid wetted by the liquid in which it is immersed, the bubble will be convex to the liquid at all points, and the curvature at a will have a greater radius than at b in the principal section. Now the normal pressure is proportional to the sum of the reciprocals of the radii of curvature in the principal sections, and we shall have the pressure at bb greater than at a when the radius of curvature at b in the principal plane represented in the figure is less than half that at a. In this case the bubble will be driven towards the point and its escape facilitated.



The thinning out of the adhering film at a point, or sharp edge of a solid, must facilitate the rolling up of the film into bubbles when the solid is immersed in a fluid which has in contact with the solid a smaller surface tension, since it affords good opportunity for the second fluid to wet the solid.

If the liquid contain in solution some gas G, other than air, this gas will evaporate into the bubbles of air until the pressure of G in the bubble attains a certain maximum depending on the temperature and on the amount of G dissolved in the liquid. The escape of the bubbles will be hastened by their expansion and they will remove more or less of G from the liquid under circumstances of temperature and pressure which would not allow of the independent formation of bubbles of G within the liquid. For if we consider that the normal pressure on a bubble of G in a liquid is proportional to the tension of the common surface of the liquid and G, and inversely proportional to the radius of curvature of the bubble, this pressure will be enormous when the bubble is very minute and must in general cause any very small bubbles, if such should form, to be immediately re-dissolved. If G, instead of being a gas merely dissolved in the liquid, be a product of chemical decomposition, either of the liquid itself or of something dissolved in it, this action which prevents the separation of gas within the liquid must have the same effect as an increase of pressure in preventing decomposition within the body of the liquid. The effect of pressure in increasing the stability of compounds which give gaseous products of decomposition is well known. If then we have a solution, say, of hydrogen peroxide, and some of it decompose, as it always tends to do, within the body of

the liquid, it will decompose into water and a solution of oxygen, as distinct from gaseous oxygen. From analogy we might expect this decomposition to be limited by the amount of oxygen in solution. Water dissolves a certain amount of oxygen, and so long as it does so without the help of external energy there will be a dissipation of energy in the action. When the water is "saturated" it requires an external source of energy, such as is supplied for instance by an increase of pressure, to make it dissolve more. We might expect then that the spontaneous decomposition of hydrogen peroxide into water and a solution of oxygen would cease as soon as the work needed to make the water dissolve more oxygen exceeded the availability of the store of energy in the hydrogen peroxide. If the amount of oxygen brought into solution in this way exceeded the quantity required to "saturate" the water, at the particular temperature and pressure to which it is subject, a slow evaporation of oxygen from the surface would go on, and a gradual spontaneous decomposition of the peroxide. This would be much increased if the surface at which the evaporation could occur were increased by the introduction of bubbles of air into the interior and the passage of such bubbles through the liquid. Such action may be sufficient to account for the slow decomposition of a neutral solution of hydrogen peroxide which occurs in glass vessels. Each bubble of gas seems to form at the surface of the glass and as it rises through the liquid visibly expands taking up oxygen as it goes. But it is insufficient to account for the very different degrees of activity shewn by different substances in promoting decomposition; and for the very remarkable effect of a minute quantity of acid in increasing the stability of the compound.

The action by which solids condense on their surfaces films of the gases in which they are immersed occurs with liquids too. The changes in the surface tension of mercury, used as an electrode, according to the nature of the gas disengaged upon it are remarkable enough. Also the miscibility of two liquids depends on the surface tension between them. Quincke has laid down the proposition that if the surface tension between two fluids be zero they are miscible in all proportions, and no drops or bubbles of the one form in the interior of the other. Moreover it appears that the surface tension between them is the smaller according as the fluids are more miscible (Pogg. CXXXIX. 87). The solution of one fluid in another becomes then closely connected with, even if it be not a result of, capillary action; and the capillary constant for a mixture is well known to be different, in general, from the constant for either component.

We may compare the solution of one fluid A in another B to a wetting of the internal surface of B by A. Indeed if distilled water containing air in solution be mixed quickly with strong

alcohol the whole liquid becomes filled with little bubbles of air which rise, not from the surface in contact with the containing vessel, but, from all parts of the interior of the mass. It looks as if the surface tension between water and alcohol being much less than between water and air, the air was rolled up into bubbles in all parts of the liquid, just as it is rolled into bubbles on a metal plate wetted with water. If a mixture of alcohol and water is mixed with a very strong solution of potassium carbonate the liquid becomes turbid with drops of alcohol which are formed exactly in the same way as the bubbles of air. The surface tension between water and the solution of potassium carbonate is less than between alcohol and the solution of potassium carbonate and the alcohol is rolled into drops according to the laws of capillary action.

Now the surface tension of water containing oxygen in solution will, in general, be different from what it is when the water is free from oxygen. Moreover it will be different, in general, where it is in contact with different substances; it may be greater for some and less for others. Also the presence of even minute quantities of acid or alkali will sensibly affect it, at least where it comes in contact with some substances.

Now surface tension is a form of potential energy, and by the laws of mechanics the several substances coming in contact must tend to arrange themselves so that this potential energy may be a minimum.

This principle appears to me to be one of wide application in chemistry, but hitherto neglected. Whenever the products of a chemical change are such as will produce a diminution of surface tension, there is a tendency towards that change; and this tendency will become effective if, on the whole, the transformations of energy by the alterations of surface tension and of chemical combination result in a degradation. Hence if we have a solution in water of some substance A in contact with some solid S, then if the surface tension between the solution and S is diminished by increasing the proportion of A in the solution, there will be an excess of A drawn into the film in contact with S. On the other hand, if pure water have less surface tension than the solution of A where it is in contact with S there will be an expulsion of A from the film in contact with S. This is not pure speculation. We have tolerably conclusive proof that this kind of action does take place. It is an old observation that when an aqueous solution of acetic acid is filtered through clean quartz sand the liquid which first passes through is almost free from the acid, and it is only when the sand has got a film of acetic acid on its surface that the mixture passes through unaltered. A similar action of quartz sand on a mixture of ethylic alcohol and fousel oil has also

been observed. Charcoal filters doubtless act in the same kind of way, though the chemical changes which occur in the films on charcoal are not so easily analysed.

Suppose now that we have a platinum plate immersed in water containing oxygen in solution, then if the surface tension between platinum and the solution will be diminished by increasing the quantity of oxygen in solution there will be an attraction of oxygen into the film in contact with the platinum from the rest of the solution, and the film may acquire so much oxygen as would supersaturate it if it were not close to the platinum. Further, if the water contain hydrogen peroxide in solution the oxygen to saturate the film may be drawn from the peroxide, and the decomposition of a part of it may be thereby determined. Moreover, so long as a supply of oxygen is forthcoming from more, as yet undecomposed, peroxide it seems to be possible that the movement of diffusion, even of the liquid forming the film, may still go on. Besides the heat due to the decomposition of the peroxide will produce convection currents which will help diffusion in carrying some of the supersaturated solution into the body of the liquid. As soon as the supersaturated solution is removed from the sphere of the molecular action of the platinum, which according to Quincke has a radius of not more than $\frac{1}{1000}$ millimetre and probably much less, the excess of oxygen will tend to roll up into bubbles, since by this arrangement the surface tensions will on the whole be a minimum. These bubbles will be formed quite close to the platinum and will appear to rise from its surface.

Again, we know that the surface tension of a liquid is sometimes greatly modified by the admixture of a comparatively minute quantity of some other liquid, and it is very probable that the addition of an acid, or of an alkali, to water may seriously affect its surface tension in contact with solids, not only with metals but also with glass. If the presence of acid impedes the liberation of oxygen it will also impede the decomposition of hydrogen peroxide, while if an alkali facilitates the liberation of oxygen it will also facilitate the decomposition of the peroxide.

At present there are no data by which to solve the question whether acid and alkali do impede or facilitate the liberation of oxygen from its solution in water, but it is only some such explanation which seems adequate to account for the facts observed. A metal so unalterable as gold induces decomposition of hydrogen peroxide in a neutral solution, but the addition of only a few drops of acid instantly stops the evolution of gas, while the addition of alkali hastens the streams of bubbles. It has been asserted that acids form chemical compounds with the peroxide, but there is no indication that this is the case, and the effect of the acid in stopping the escape of gas is quite evident when the

quantity of acid is so small as to be not nearly equivalent to the amount of peroxide in the solution.

I have made no special mention of the action of those compounds which, like silver oxide, themselves undergo decomposition while they induce the decomposition of the peroxide. In all these cases there is energy enough rendered available by the decomposition of the peroxide to bring about the other decomposition.

It becomes now a question of interest whether the measured variations in the surface energies of water and aqueous solutions are quantities comparable in magnitude with the work which this theory requires to be done by them. I know of no measurements of the changes of surface energy of water produced by the solution in the water of any gases except hydrochloric acid, but Quincke has measured the surface tensions of pure water and of a solution of sodium hyposulphite, and though we cannot assume that the difference between them gives any indication of the difference of surface tension of water and solutions in any other case, it will serve to give us an indication of the order of magnitude which we may expect such differences to shew.

Now Quincke's measures give a difference of surface energy per square centimetre for water and solution of hyposulphite amounting to 3·5 milligrammes. This will produce an increase of pressure upon a bubble of oxygen in the liquid of about $3{\cdot}38 \times 10^{-6}$ of an atmosphere per square centimetre. The work done by this on a bubble containing 1 cc. of oxygen would be about equal to that of a pressure of one atmosphere on a square centimetre working through $3{\cdot}38 \times 10^{-6}$ cm. Now the volume of oxygen under a pressure of one atmosphere absorbed by 1 cc. of water at 10° C. is 0·0325 cc.; and if we assume that the radius of molecular action which gives rise to capillary effects is $\frac{1}{10000}$ of a millimetre, the volume of oxygen absorbed by a film of water of that thickness extending over one square cm. will be only $3{\cdot}25 \times 10^{-7}$ cc.

The work which would have to be done in order to compress n times as much oxygen originally at atmospheric pressure into that space, with an isothermal arrangement, would be equal to the pressure of one atmosphere on one square cm. acting through $n \log n \times 3{\cdot}25 \times 10^{-7}$ cm.

Now the work above found for the change of surface energy of water, due to the solution of hyposulphite in it, is nearly equal to this when $n \log n = 10$, which gives a value for n between 5 and 6. This is as much as to say that the change of superficial energy of water in contact with air when hyposulphite is dissolved in it is capable of doing the work requisite to make the superficial film take up 5 or 6 times as much oxygen per unit of volume as the bulk of the water absorbs.

I think this shews that I am dealing with quantities which are

comparable in order of magnitude, and I do not mean that it shews anything more.

The difference between the surface energy of water in contact with mercury, and of a solution of hydrochloric acid in contact with mercury, is nearly twelve times as great as the difference which I have taken above for my illustration, so that I have not taken an exceptionally favourable case.

Charcoal is by no means a pure chemical substance, and small differences in its composition according as it is made from animal or from vegetable tissues may well give it different surface tensions when in contact with the same bodies. Such different surface tensions will, I think, suffice to account for the different activities of animal and vegetable charcoal when used as filters.

I have not attempted to trace the influence of capillarity in the chemical changes induced in substances thus adherent to charcoal, vegetable mould, and other filtering materials, though it seems probable that it may play an important part in bringing about such changes. It may assist materially in the processes of secretion, and in the rapid exchange of oxygen for carbonic acid gas which occurs in respiration; and it can hardly fail to have its influence in many of the chemical changes which occur in the organs of plants and animals.

Further, it does not appear to be in any way necessary that the mutually reacting bodies should either of them be a fluid. For solids have their surface energies and must influence one another when sufficiently approached. In this way we may explain the action of manganese dioxide in promoting the disengagement of oxygen from potassium chlorate at temperatures below the fusing point of the latter.

That such disengagement does occur before the potassium chlorate enters into fusion I take on the authority of Wiederholt and others, though in general, so far as my own observations go, the decomposition of the chlorate is always accompanied by fusion of the part decomposed.

That solution is the spreading of a film of the dissolved substance over the internal surface of the menstruum has been, I believe, suggested before. No one has as yet attempted to work out the evidence for or against such a supposition. The problem however is a very interesting one, as it is well known that the solution of substances, and their separation from solution, have considerable influence upon chemical changes. The supposition seems to offer an obvious explanation of the deposition of crystals upon nuclei and upon surfaces from which the adhering films have been partly removed by rubbing.

(2) *On Homotaxis.* By J. E. Marr, M.A.

A. *Introduction of the term.* The difficulty of proving the contemporaneity of rocks by their included fossils was shewn by Herbert Spencer in an article upon "Illogical Geology," reprinted in his "Essays." At a later date, Professor Huxley, in his Anniversary Address to the Geological Society, in 1862 (reprinted in his "Lay Sermons"), insisted on the danger of confusing correspondence in succession with correspondence in age, and proposed the term "Homotaxis" (similarity of order), to express the similarity of serial relation of the faunas of strata of different areas. He observes that "whether the hypothesis of single or of multiple specific centres be adopted, similarity of organic contents cannot possibly afford any proofs of the synchrony of the deposits which contain them; on the contrary, it is demonstrably compatible with the lapse of the most prodigious intervals of time, and with interposition of vast changes in the organic and inorganic worlds, between the epochs in which such deposits were formed."

Again, "there seems, then, no escape from the admission that neither physical geology nor palæontology, possesses any method by which the absolute synchronism of two strata can be demonstrated. All that geology can prove is local order of succession. It is mathematically certain that, in any given vertical linear succession of an undisturbed series of sedimentary deposits, the bed which lies lowest is the oldest. In any other vertical linear succession of the same series, of course, corresponding beds will occur in a similar order; but however great may be the probability, no man can say with absolute certainty that the beds in the two sections were synchronously deposited. For areas of moderate extent, it is doubtless true that no practical evil is likely to result from assuming the corresponding beds to be synchronous or strictly contemporaneous; and there are multitudes of accessory circumstances which may fully justify the assumption of such synchrony. But the moment the geologist has to deal with large areas, or with completely separated deposits, the mischief of confounding that 'homotaxis' or 'similarity of arrangement,' which *can* be demonstrated, with 'synchrony' or 'identity of date,' for which there is not a shadow of proof, under the one common term of 'contemporaneity' becomes incalculable, and proves the constant source of gratuitous speculations."

The danger of identifying widely separated strata as actually contemporaneous is of course due to the certainty, that, whether originating in single or multiple specific centres, they must have migrated outward from these. Consequently, if we have two deposits *A* and *B* remote from one another, containing the same dominant forms of life, and each capable of being split up into

subdivisions, characterised by the occurrence of some dominant species in each, which we may call a, b, c, ..., the subdivisions of the two deposits may and probably will closely correspond with each other, as regards fossil contents, and yet the forms of the group a may have originated in the area of A, whilst other forms existed in B, and by the time the forms of the group a have migrated to the area of B, the forms of the group b may have come into existence in A, and so on.

Admitting that Professor Huxley's warning was one which geologists required, it seems to me that many writers have been so impressed by it, that they have practically come to look upon the value of palæontological evidence as a means of comparing deposits, with extreme suspicion, and I propose, therefore, in this communication to inquire whether this suspicion is justifiable, and if so, to what extent. In so doing, I shall consider chiefly the evidence afforded by the graptolites, not that this seems to me to be essentially different from that yielded by an examination of any other group of organisms, but firstly, because our knowlege of these forms is, thanks chiefly to the researches of Prof. Lapworth, very considerable, secondly, because I have paid more attention to these than to other life-forms; and thirdly, because there is perhaps greater reluctance to abide by the decision of the palæontologist in the case of the older rocks in which this group occurs, than in that of more modern deposits.

B. *Consideration of the facts of distribution.* In order to lay before you the importance of the results achieved of recent years by a study of the Rhabdophora or graptolites, I shall have occasion to refer largely to Prof. Lapworth's very important paper "On the Geological Distribution of the Rhabdophora."* In this paper he shews that the Lower Palæozoic rocks contain a series of graptolitic faunas, which are marked by the constant association of certain forms, with some dominant form. To prevent overcrowding of detail, I give a list of such dominant forms, in the order in which they always appear, indicating at the same time the areas in which they have been discovered. Beginning with the oldest forms, we have

Dictyograptus	in Britain, Belgium, Scandinavia, America.
Bryograptus	— Britain, Scandinavia.
Phyllograptus	— Britain, Scandinavia, America, Australia.
Murchisoni-form	
Didymograptus	— Britain, France, Portugal, Scandinavia.
Cœnograptus	— Britain, America, Australia.
Amphigraptus	— Britain, America.

* *Ann. and Mag. Nat. Hist.* Ser. v. Vol. III.

Dimorphograptus in Britain, Thuringia, Scandinavia.
Rastrites — Britain, Thuringia, Bohemia, Scandinavia.
Cyrtograptus — Britain, Bohemia, Scandinavia.

It will be seen that the order of succession is constant.

Nowhere do we meet with Cyrtograptus in beds below those containing Rastrites, and nowhere with beds containing Rastrites below those with Dimorphograptus, and similarly with the other forms.

Again, not only is the order of appearance identical, but in many cases, if not in all, that of disappearance of the forms mentioned, and whether this be the case or not, the genera disappear in different places at a time when those places were marked by the occupation of similar faunas. For instance, Rastrites disappears at the period when the areas in which it is found were occupied by a fauna which Prof. Lapworth has described as the Gala fauna.

The major subdivisions I have marked out by the occurrence in each of one dominant form, can be split up into a series of zones, each of which has a characteristic assemblage grouped around one dominant form. To illustrate this, I will quote one instance, that of the correspondence of the zones of the Birkhill shales of the south of Scotland, as worked out by Professor Lapworth, with those of the similar beds of the south of Sweden, described by the late Dr Tullberg. The zones are as follows, in ascending order:

S. Scotland.		S. Scandinavia (Scania).
Zone of Diplograptus acuminatus	=	Zone of Diplograptus, n. sp.
—— Diplograptus vesiculosus	=	—— Monograptus cyphus
—— Monograptus gregarius	=	Zones of { Monograptus gregarius, Monograptus convolutus
Zones of { Cephalograptus cometa, Monograptus spinigerus }	=	Zone of { Cephalograptus cometa
Zone of Rastrites maximus	=	Zone of Monograptus turriculatus.

The thickness of the whole of the Birkhill shales is only about 140 ft., that of the corresponding Rastrites beds of Scania about 400 feet. Comparing these zones in greater detail, we find that (i) the lowest zone in Sweden (which is placed by Dr Tullberg with the group below the Rastrites beds), is marked by the absence of Monograptus, which first appears in the next succeeding zone, and by the presence, along with the dominant form, of Climacograptus scalaris, which passes up into succeeding zones. In the south of Scotland, along with the dormant form, we find the same Climacograptus and a Dimorphograptus, and here also Monograptus is absent, but appears in the succeeding zone. (ii) In the Mono-

graptus cyphus zone of Sweden, occurs also a Dimorphograptus, a Diplograptus and Climacograptus scalaris, and Dimorphograptus dies out here. In Scotland, Monograptus cyphus is not recorded from this zone, but Dimorphograptus disappears here. (iii) In the two succeeding zones in Scandinavia, we find, (*a*) Monograptus triangulatus, M. fimbriatus, Rastrites peregrinus, &c., and (*b*) Monograptus convolutus, M. lobiferus, M. communis, M. leptotheca, Rastrites peregrinus, and Cephalograptus folium. This is the first appearance of Rastrites. Each one of these forms occurs in the gregarius zone of Scotland, along with many other species, and here again we mark the first appearance of Rastrites. (iv) The zone of Cephalograptus cometa in Sweden contains also Monograptus spinigerus, M. intermedius, M. Clingani, M. lobiferus, M. argutus, Diplograptus Hughesii, Cephalograptus cometa. Each of them, save M. intermedius, is recorded from the two zones in Scotland which correspond with this Swedish zone. (v) The zone of Monograptus turriculatus in Sweden contains also Monograptus crispus, but Rastrites maximus has not been detected in it. Monograptus turriculatus however occurs in this zone in Scotland.

The correspondence is very complete considering that our knowledge of these beds has only been obtained comparatively recently, and Prof. Lapworth informs me that the Birkhill shales might be further subdivided, for he paid attention to the lithological character of the beds as well as their fossil contents when working out the S. Scotch succession, and did not therefore trouble himself with making more minute subdivisions. As it is, a detailed examination of the corresponding beds in the Lake District has enabled me to make a very close comparison both with the Birkhill shales of Scotland, and the Rastrites beds of Sweden, which I hope shortly to give an account of elsewhere.

Other important facts may be gathered from an examination of the lithological characters of the graptolitic deposits. I will consider the case of the rocks containing the Birkhill, Gala, and Riccarton faunas of Prof. Lapworth. The beds containing the fossils are usually dark shales, but they present considerable variations. The Birkhill beds of the Lake district consist mainly of black or very dark shales, the Gala beds of the same area mainly of pale green bands, and the Riccarton beds of grey flaggy rocks.

It might be supposed that the variations in the characters of the faunas was actually produced by variations in the nature of the sediment, but to prove that this is not the case, we find in Bohemia a succession of black shales, closely similar in general appearance, and nevertheless the succession of Birkhill, Gala, and Riccarton faunas is found exactly as in Britain. Other cases might be given, but one is sufficient for illustration.

Again, it is very noteworthy that the change in the character of the faunas is accompanied by change in what may be termed accidental characters of the rocks which contain those faunas,—accidental inasmuch as it is difficult to see how these characters could produce any effect upon the organisms. Taking the same three faunas, we find that the shales containing the Birkhill faunas, are usually marked by a comparative paucity of light green bands, whereas the Gala beds contain an abundance of such, as seen not only in Britain, but also in Sweden and Bohemia. As the graptolites are generally absent from these green bands, and occur in the dark shales interstratified with them, it is hard to understand how their relative abundance can produce any effect upon the faunas. Still more marked is the occurrence of large elliptical concretionary nodules in the beds containing the Riccarton fauna, in Britain, Scandinavia, France, and Bohemia. These nodules are never, so far as I am aware, discovered in the rocks containing the Birkhill and Gala forms of life.

So much for the facts presented by an examination of the graptolitic deposits. Before discussing these, I may call attention to two cases of non-graptolitic rocks. In the Lake district, a band of rock, the Coniston Limestone, is succeeded by a thin limestone, which I have spoken of as the Staurocephalus zone, and this in turn by black shales. This Staurocephalus limestone is only a foot or two in thickness, nevertheless a similar fauna to that which it contains, is found in a similar position in Wales, Scotland, and Scandinavia, and probably elsewhere. The lithological characters of this band, which are peculiar, are also remarkably constant. It is noticeable also, that although in the Lake district it reposes upon a limestone, and is succeeded by shales, its fauna is entirely different from that of the underlying limestone, and closely similar to though richer than that of the succeeding shales. This seems to indicate that the change in the fauna was not produced by a change in the supply of sediment.

The close similarity of the Ammonite zones of the Jurassic rocks in Britain and Germany is too well known to need illustration.

C. Difficulties in the way of abandoning the view of contemporaneity. These are two-fold. Firstly, those connected with the migration of the organisms. It has been seen that certain dominant genera appear in like order over wide areas, and that speaking generally they disappear in like order also. This is only conceivable upon the supposition that they originated in the same region, and that in each case the direction of migration was the same, which is very unlikely, or that the time taken for dispersal was short, as compared with the time during which each genus

was in existence, which is more probable. It has been observed furthermore, that a similar resemblance may be traced with species, even in the case of thin zones of rock, and the alternatives are the same in this case also.

Secondly, although the change of faunas is not necessarily accompanied by a complete change in the lithological characters of the rocks containing them, it is frequently so marked, and even if not, is very often accompanied by a change of what I have spoken of as the accidental characters of the rocks. This is also traceable to some extent, in comparing thin zones, as I hope to shew elsewhere. If the change of lithological character does not produce the change of organisms, and I have given reasons for supposing that it does not, we are compelled to conclude that the organic and lithological changes took place simultaneously in one area, and (according to the Homotaxis theory) later on occurred also simultaneously in another. In any case it points to the conclusion that lithological and organic change were due to one and the same cause, which cause may have acted simultaneously over wide areas, nevertheless, one objection which has been made to the correlation of deposits as synchronous, is founded upon the migration of faunas, owing to lithological change.

D. Possibility of ascertaining practical contemporaneity of beds over wide areas. We have been driven to admit either that groups of species originated after each other in one and the same area, and migrated thence along the same lines time after time, or else that the time taken for dispersal was short, as compared with the time of the duration of each group of species. But as in some cases, the groups of species are limited to zones a foot or two in thickness, the time taken for dispersal was short as compared with the time necessary for the accumulation of the sediment of these zones. In other words, the amount of sediment accumulated in these cases during the period of dispersal of the organisms was so thin, that it may be practically neglected, and the zones spoken of as contemporaneous. Thus though we admit that a film of sediment may have been laid down on the sea-floor of the S. Scotch area during the formation of the rocks of the gregarius-zone of the Birkhill shales, before it was accumulated in S. Sweden, it is hard to resist the conclusion that these two zones were being formed simultaneously in the two areas during most of the period when their characteristic organisms existed. If this be true of these minor subdivisions, it is much more true of the major ones, of which they form but a fraction.

E. Difficulties in the way of admitting synchronism of formation. It may be objected that the cases upon which I have laid most stress are cases of exceptional slowness of accumulation of sediment,

and such is indeed the case. For example, Prof. Lapworth has shewn that the 140 feet of Birkhill shales of the Moffat district are represented in the Girvan area by over 1000 feet of sediment, (*Q. J. G. S.*, Vol. XXXVIII. p. 537). The result of this thickening out is not to confuse the zones, but to cause the fossils to be distributed through a greater thickness of beds. There is no practical difficulty raised in the case of sediments of greater thickness than those we have specially considered. The effect will be to cause the geologist to subdivide the rocks into zones of greater thickness, rather than to make him attempt to speak of deposits as synchronous which are in reality not so. It is found also in actual fact, that the quickly formed deposits are those which rapidly change their characters when traced laterally, whereas the slowly-formed ones have a wider distribution without much alteration. In other words the slowly-accumulated deposits were wide-spread horizontally, the quickly-formed ones were local. This is the case at the present day. In attempting to correlate the rapidly-accumulated deposits of widely distant areas, one meets not only with changes of lithological character, but also with differences in the faunas, so that there is no temptation to insist upon the synchronous formation of different portions of these deposits. I may make this clearer by an example. Suppose we have in two widely separated areas, thin deposits of A age divisible into zones a, b, c..., and these zones are similar in the two areas. Above these deposits we meet with thick deposits of B age, which are mainly grits in one area and calcareous ashes in the other, and that these are succeeded by deposits of C age in the two areas, which as in the case of those of A age are divisible into zones x, y, z..., which are similar in the two areas: then, although we have proof that the B group are really as a whole contemporaneous, we shall be unable to compare their subdivisions to any extent, for we shall find a succession of zones of sand-loving organisms 1, 2, 3 in the one case, and of another set 1′, 2′, 3′ having a different habitat in the other.

The case of reappearances must be considered. It is well known that fossils occur in certain bands of the Bohemian Lower Palæozoic rocks, which appear to be absent from intermediate bands. Barrande divides his stage D into 5 bands, of which d_1, d_3, d_5 have a general resemblance to one another, and differ from d_2 and d_4 which are in turn somewhat alike. In d_1, d_3 and d_5 we find Æglina rediviva and Dionide formosa, whilst the genera Areia, Carmon, Lichas, Ogygia, and Proetus occur in d_1 and d_5 and not in the intermediate bands. In d_2 and d_4 are Illænus distinctus and I. transfuga, which are absent from the other bands. Barrande shews a similar set of recurrences of fossils from g_1 in g_3 and of g_1 in h_1. These examples seem to throw doubt upon the possibility

of correlation. It must be remembered however that they are not dominant forms, and that there is no difficulty in distinguishing d_2 from d_3 and so with the other bands. Moreover it is not even necessary to imagine migration from the area during the period of accumulation of the intervening sediment, though very probably this did occur. We do find some forms which are abundant in two beds and which are very rare in the intervening ones, and it is possible that in the cases above quoted, the forms struggled on in the area under unsuitable conditions, and again increased in numbers when the conditions became once more favourable. In this case they might well escape detection in the intervening beds, even when the beds were so thoroughly worked as were those of the Bohemian basin by Barrande.

F. Possible cause of the change of graptolitic faunas. It has been seen that change of lithological character does not necessarily produce change of fauna, though the two do frequently accompany one another, and therefore seem to be produced by some common cause. This cause is either physiological or physical. The coincidence of a complete change of facies of the organisms with that of the containing rocks seems to point to the latter. Yet we find in the graptolitic bands, that thin pale green bands occur which are identical in lithological character with the black graptolitic bands, save for the colour and the great rarity of graptolites contained in them. The colouring matter of the black bands is certainly organic, and seems to be due to the graptolites themselves. Yet these pale green bands are very constant. One of them, a quarter of an inch thick, occurs at precisely the same horizon in the Lake District for a distance of over half-a-dozen miles, and is not separated from the black graptolitic bands above and below by any plane of stratification. As it appears that this deposit is quite like the rock in which it lies, except for the absence of graptolites, we must look for some cause which would cause the graptolites to disappear from the area, or to linger on under great disadvantages, without altering the character of the sediment. The only cause I can think of is change of climate; I suggest this as a possible explanation, without insisting on its probability. If such climatic changes occurred, and are accountable as Dr Croll thinks for variations in the character of Carboniferous deposits, they are equally likely to have produced similar effects here. In this case, all the difficulties of synchronism are overcome, and we see a reason for believing what the evidence certainly supports, viz.: the possibility of successfully correlating deposits of remote areas.

G. Conclusion. An examination of the facts seems to me to indicate that we need not be frightened of attempting correlations

of the rocks of different areas. In the cases where the similarity of organic contents of two sets of deposits "is demonstrably compatible with the lapse of the most prodigious intervals of time, and with interposition of vast changes in the organic and inorganic worlds, between the epochs in which such deposits were formed," the geologist would have no temptation to try the correlation of those deposits, as they would form small portions of similar deposits containing the same organisms throughout, and would be found to die out rapidly when traced horizontally.

My purpose in bringing forward this paper is to urge the students of the University to accumulate details in order to test further the views I hold as to synchronism. A few years ago, the very deposits I have been using chiefly by way of illustration were cited as instances of the danger of drawing conclusions as to contemporaneity. Now, as Prof. Lapworth says (Geological Distribution of the Rhabdophora), "the host of proofs supposed to be afforded by the abnormalities of the vertical distribution of the Graptolithina, in favour of the doctrines of migration and colonies, vanish into thin air. These apparent evidences are now seen to have been fallacious appearances, due simply to defective knowledge." But many difficulties still remain. I have not alluded to the many conflicting evidences between terrestrial faunas and floras and marine faunas, cited by Dr Blanford in his presidential address to section C. of the British Association at Montreal, nor to the occurrence of an intermixture of Permian and Lower Carboniferous marine organisms in Spitzbergen which are stated to be below beds of Coal measure age (cf. Nordenskjöld, *Geol. Mag.* 1876 not because I have ignored them, but because I believe that we should suspend our judgment until further information has been obtained concerning them. Admitting that the supposed admixture is not due to errors in the interpretation of the stratigraphy in the different cases, they do not outweigh the great amount of cumulative evidence in favour of correlation of beds in widely remote areas; nevertheless, still more evidence must be gathered not only to convince those who have not made a special study of the geological distribution of organisms, but also in order to determine the laws which govern that distribution, and every discovery of new facts is a distinct step toward this important end.

(3) *Note on the functions of the secreting hairs found upon the nodes of young stems of Thunbergia laurifolia.* By WALTER GARDINER, M.A.

In a paper by W. Gardiner and R. I. Lynch, read before the Society on Nov. 10, 1884, special attention was drawn to the cup-shaped secretory hairs of Thunbergia laurifolia. The secretion

was shown to be of a watery nature, and to possess a slightly acid reaction. The function performed by the hairs and their secretion was at the time uncertain. The author of the present paper believes that the secretion serves to attract ants, which besides feeding upon it, also protect the thin young climbing shoots by attacking and destroying other creeping insects of alien race with whom they may meet in their passage up and down the stem. Many of these insects such as cockroaches and caterpillars are known to be very destructive to young buds. The author has been able to establish from actual observation of plants at Kew, that ants actually visit the hairs and feed upon the secretion. Similar hairs are found upon the calyx. In a certain unnamed species of Combretum sent by Ernst from Caracas hairs of essentially the same structure were also observed.

(4) *On the petiolar glands of the Ipomœas.* By WALTER GARDINER, M.A.

In all the species of Ipomœa which were examined, secretory structures probably of the nature of extra-floral nectaries were observed. A pair of such glands are found in each leaf; one on either side of the petiole at the point of junction of the petiole and lamina. The leaves of Ipomœa Horsfalli possess glands of an especially simple structure while those of Ipomœa paniculata exhibit a distinctly complex organisation resembling in fact a racemose gland and possessing a well developed duct. Other species present gradations between these two extremes. The secretion arises from capitate hairs which are either situated singly at the bottom of a depression of the epidermis, or in numbers line a saccate or racemose involution of the same. The author believes that the secretion of these petiolar glands attracts ants, which in their turn serve to protect the plant.

(5) *On the occurrence of secreting glandular organs on the leaves of some Aroids.* By WALTER GARDINER, M.A.

The author remarked that it has been frequently stated that the entire absence of all extra-floral secretory structures in Monocotyledonous plants furnishes one of the most striking points of difference between the above-named group and the Dicotyledons. One would be led to expect however that some form of secretive organ should be present, and that probably they would be found—if anywhere—among the Aroids. Guided by these considerations the author made a careful examination of the Aroids at Kew, and was so fortunate as to find two individuals, viz. Aglaonema Mannii and Alocasia cuprea, which appear to him to possess definite organs of secretion. The structure of these organs was then shortly

described, and a comparison was instituted between them, and certain forms of extra-floral nectaries. As to the existence of intramural glands e.g. in Anthurium punctatum the author's observations confirmed those of Dalitzch recently published in the Botanisches Centralblatt.

February 28th, 1887.

MR TROTTER, PRESIDENT, IN THE CHAIR.

MR A. E. H. LOVE, B.A., St John's College, was elected a Fellow.

The following Communications were made:

(1) *Experiments on the magnetization of iron rods, especially on the effect of narrow crevasses at right angles to their length.* By Prof. J. J. THOMSON, M.A., and H. F. NEWALL, M.A.

IN some experiments on which we have lately been engaged it was necessary to use a very strong magnetic field, and in measuring its strength we arrived at results which seemed worth following up.

Firstly, we found that we were dealing with magnetic inductions very much larger than any yet recorded. And secondly, we found that the strength of magnetization of an iron rod was enormously reduced by simply cutting it in two across the lines of force, and putting the cut ends together, separated by a very small interval, the reduction rising to between 20 and 40 per cent. when the separation was only a small fraction of the diameter of the rod.

The present note gives the result of some further experiments we made on these points.

Experiments on very strong magnetic fields.

During last autumn we worked with two large coils each having 9 turns of stout copper wire per centimetre of their length, the wires being large enough to carry a current of more than 120 ampères, so that we could produce a magnetic field whose strength was as large as 1500 C.G.S. units. The total length of the coils was 40 cms. They were separated by an interval of 15 cms. The current was supplied by 10 storage cells, generally in series, and was measured by a graded ammeter of Sir William Thomson's form.

Iron cores were sometimes made of bundles of soft Swedish iron wires held together in a brass tube, sometimes of solid Low Moor

iron wire put into the coils, the length and diameter of the cores being varied considerably.

The magnetic induction was measured by the ballistic method. One turn of insulated wire was wrapped round the core at its middle and the ends connected with wires leading to a ballistic galvanometer placed at a distance of 9 metres. The value of the deflections was obtained by discharging a known quantity of electricity through the galvanometer: for this purpose we used a condenser of known capacity charged by Clark cells. In later experiments the value of the galvanometer deflections has been further checked by observing the deflection produced by suddenly turning a coil of known area through 180° in the earth's magnetic field, sometimes round a vertical axis and sometimes round a horizontal one, this coil being in the galvanometer circuit. We have got with no special effort a magnetic induction of 28,000 C. G. S. units when the magnetising force was about 1200, and are led to believe that considerably higher numbers might be reached if longer coils were used, and the ratio of the length of the core to its diameter were increased. The highest induction yet recorded is 21,000 by Mr Shelford Bidwell, and Prof. Rowland gives 17,500 as the probable maximum attainable for iron. We have no reason to believe that with our present apparatus this is the limit we can obtain, but as we learn from Prof. Ewing that he is working at the same subject and has got inductions as high as 33,000, we have not pursued the matter further.

For our main experiment however, the cores had to be cut across in the middle and separated by a small interval. We found that the crevasse thus formed reduced the magnetization to a most surprising degree, and we will now proceed to describe our further experiments on this point.

Experiments on the effects of cutting iron cores transversely to the lines of force.

With the above described coils the induction was measured for a core of solid soft iron (Low Moor) of length 170 cms., about 5 ft. 6 in., and diameter 3·2 cms., about $1\frac{1}{4}$ in. This core was then cut into two equal parts, the ends being turned up in a lathe with no very *special* pains: the core was again put in the coils, as if to imitate the original uncut core. The induction for the cut bar was now measured and found to be more than 10 % lower than for the uncut core. If the two parts instead of touching at the middle, were separated by 1 cm. the induction was now found to be reduced by 60 %.

We have used smaller apparatus lately to inquire further into this point.

The magnetizing coil was 16 cms. long and had 140 turns per

cm. of its length, and currents of between ·5 and 3·5 ampères were passed through it, so that we dealt with magnetizing forces of between 88 and 616 C. G. S. units.

Lengths were cut from a round rod of soft iron of 12·7 mm. diameter, and were carefully softened. One length was 10 cms., two others were 5 cms., so that these two put end to end made up a single length equal to the uncut piece of 10 cms. We shall speak of the single one as the continuous core, the two 5 cms. pieces as the cut core.

To measure the induction when these cores were put into the middle of the magnetizing coil, two turns of insulated wire were put round the core, and the ends connected as before with the ballistic galvanometer. For the cut core the wire was wound as near to the end of one of the halves as could be managed, and it was tied in this position, as it was found that a slight shifting of these wires along the length of the core produced considerable effects on the inductions. The position we have given to the turns of wire, which we shall speak of as the induction coil, will always give higher results than if the induction coil had been set midway between the cut ends: so that the reductions in the induction are always understated in the results we give in the following Table I.

TABLE I.

Cores 1·27 cm. diam. 10 cms. long.

Magnetizing force due to coil ($4\pi n\gamma$)	200	280	460
	Induction.		
Continuous core	6,048	8,400	14,160
Cut core, worked ends (turned in a lathe) touching...	5,472 9		
„ „ ends (worked on oilstone) touching ...	6,000	8,352	13,920
„ „ ends separated by ·127 mm. or $\frac{1}{100}$ diam.	5,328 12	7,296 13	12,432 12
„ „ „ „ 2 × ·127 mm. or $\frac{2}{100}$ diam.	4,800 21	6,624 21	11,136 21
„ „ „ „ 3 × ·127 mm. or $\frac{3}{100}$ diam.	4,416 27	... ...	... ...
„ „ „ „ 4 × ·127 mm. or $\frac{4}{100}$ diam.	4,032 33	5,568 34	9,504 33
„ „ „ „ $\frac{20}{100}$ diam.	2,352 61	3,264 61	...
1 diam.	1,344 78	1,824 78	...

The numbers following the induction measures denote the percentage reduction due to separation. The magnetizing forces here used are not such as to 'saturate' the cores, though there are signs of near approach of this.

Tests have been applied in very varied ways to eliminate sources of uncertain error. The cores were moved into different positions in the magnetizing spiral through distances considerably greater than accidental variation; the induction coil was moved to different positions along the core—the alterations in the induction due to such changes led us to tie the induction coil firmly to the core; the cores were taken out and set afresh; the cut cores *were pressed together by means of clamps.*

Experiments on bars of different diameters.

The results recorded in Table I. were got for cores whose diameter was 12·7 mm., and of total length 10 cms. The experiments were repeated with cores of the same length but of smaller diameter, namely 3·3 mm. The cut ends were carefully worked on oilstone, and the general results showed that but little difference was made by altering the dimensions of the core: or in other words, that there was the same surprising reduction for cores of small diameter as for cores of large diameter. When the separation of the worked ends was about

·4 mm. the induction was reduced 26 per cent.
·8 mm. " " " " 42 " "
3·3 mm. " " " " 77 " "

Direction of the lines of force about the crevasse.

Cores, cut as above described, were put into a long coil having two layers of wire, and fitting closely to the cores: and iron filings

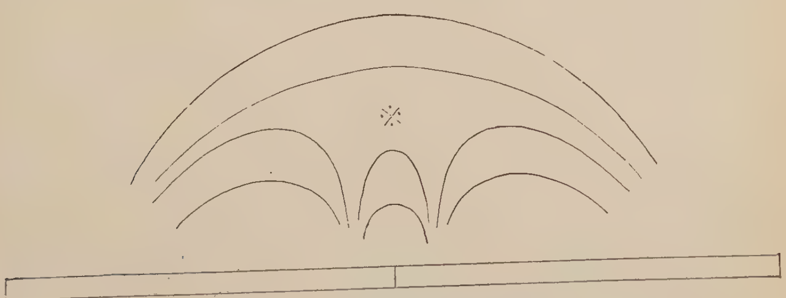


Fig. 1. Worked ends touching.

were sprinkled on paper laid horizontally in a plane through the axis of the cores. Figures 1 and 2 were traced through the iron filings.

It will be seen that there are fairly definite poles on either side

of the crevasse; that these poles are situated at some distance from the core ends, even from the roughly worked ends, when these are set touching one another; that the distance of the poles from

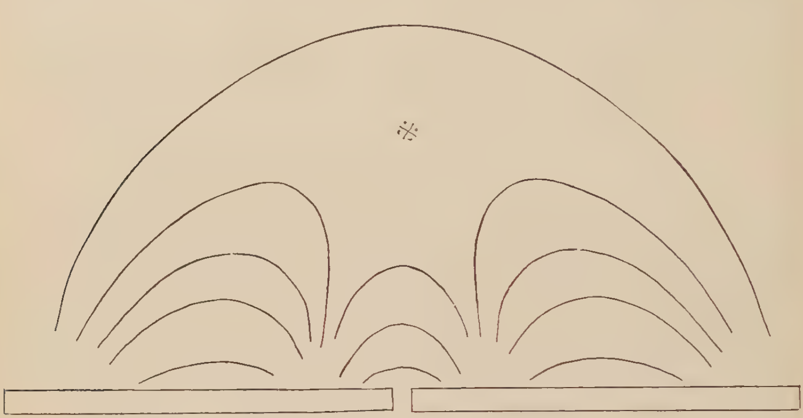


FIG. 2. WORKED ENDS SEPARATED BY ONE DIAMETER.

the ends of the cores changes with the separation of the worked ends; and that there is a neutral point (marked * in the figures) whose locus is a circle round the core in a plane perpendicular to the axis, and the centre on the axis in the middle of the crevasse.

Assuming that the external action of the cores may be taken as due to a distribution of magnetism representible by poles placed at a definite position in the cores, we may express the distance of the neutral point from the axis in terms of the distances between these poles. Consider the cut cores as replaced by four poles as in the figure 3.

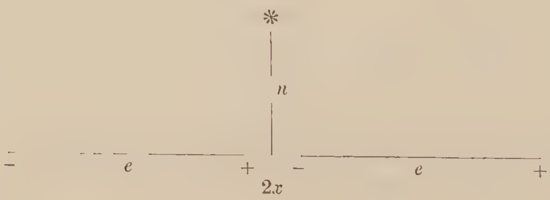


FIG. 3.

Let the separation of the middle poles be $2x$, and the distance between one middle pole and the nearest outer pole be l. Then n the distance of the neutral point from the axis is related to x and l by the equation

$$x^2 = x^{\frac{2}{3}}(x+l)^{\frac{4}{3}} + x^{\frac{4}{3}}(x+l)^{\frac{2}{3}},$$

or neglecting $x^{\frac{2}{3}}$ in comparison with $l^{\frac{2}{3}}$

$$n = \sqrt[3]{l^2 x}.$$

Hence n varies very slowly with x.

If we might consider that in our present case there are definite poles, whose strength does not vary with the separation of the worked ends, the reduction in the induction would be easily calculable as due to the spreading of the lines of force. We can trace the neutral point up to the surface of the cores, when there is no separation, that is, when the core is continuous. Now let the core be cut and the ends separated by a small interval: the neutral point moves outwards, and the induction falls: but the surprising magnitude of the fall for a very small separation of the worked ends shows that the poles must be much more widely separated.

Some light seems to us to be thrown on the reason for these surprisingly large effects by the consideration of Green's investigation on the distribution of magnetism on a cylindrical magnet placed in a uniform magnetic field—the lines of force being parallel to the axis of the magnet. If in this case all the magnetism were collected at the ends of the bar, the above results would be quite inexplicable, for we can show that if we cut a magnet and separate the two halves, the induction across a curve midway between adjacent ends of these halves, and equal to the cross section of the magnet, differs if all the magnetism be collected at the ends, from the induction through the same curve when the ends are pushed together by a quantity proportional to the cube of the ratio of the distance between adjacent ends, to the diameter of the magnet, so that the effect of the small separations which occurred in our experiments would be quite inappreciable. If however the magnetism instead of being confined to the end extends a considerable distance along the magnet, then the induction across a section half way between the ends will be diminished very much more by putting the ends apart. Now Green's investigation shows that if a cylindrical magnet be placed in a uniform field, that if M be the total quantity on one half of the magnet, the quantity of magnetism on the flat end is $\frac{1}{2}pM$, and the distance from the end of the centre of mass of the magnetic distribution is a/p where a is the radius of the magnet, and p a quantity depending on the coefficient or magnetic induction. For the forces we used p would be about 1/20, so that the magnetism at the flat end would only be 1/40 of the total quantity of magnetism, and the centre of gravity of the distribution would be 20 radii from the end; thus the magnetism is very much diffused and cannot be approximately represented by a distribution of magnetism over the flat end. This would make it possible for the poles to be separated by a considerable distance even though

the ends of the magnet were close together. We have verified by means of iron filings the result to which we have been led by this reasoning (that the distance between the poles is very large compared with the distance between the ends); we found that even when the ends of the magnets were turned upon the lathe and then pushed together that the distance between the "poles" was more than a cm.

If the coefficient of magnetic induction is small, p is very large, and most of the magnetism is on the flat end, in this case the cutting of the magnet would make little difference, so that this effect is probably much greater in proportion for iron than for any other metal.

(2) *On some measurements of the frequencies of the notes of a whistle of adjustable pitch.* By W. N. SHAW, M.A., and F. M. TURNER.

Among the anthropometric measurements proposed by Mr Galton is the determination of the limit of audibility of sound. For this purpose an adjustable whistle has been made by the Cambridge Scientific Instrument Company. A wire ·73 mm. in diameter forms a solid piston in the pipe of the whistle, and the pitch of the whistle can be altered by pushing the wire further in or pulling it further out. The distance between two parallel discs, one fixed to the sliding wire and the other to the pipe, is measured by means of a graduated wedge, and gives directly the length of the pipe of the whistle.

In order to test the sensibility of a person's hearing, the pitch of the note is gradually raised by pushing in the wire piston until a limit is reached beyond which the sound is inaudible, and the distance between the discs is then measured and gives the length of the pipe of the whistle. The sound is produced by suddenly squeezing between the thumb and finger a small indiarubber bladder attached to the whistle.

From the measurement thus obtained we can deduce the frequency of the highest note audible by a particular subject if we assume (1) the velocity of sound in air at the time of observation, (2) that the wave-length of the note in air is four times the length of the pipe. In that case if l be the length of the whistle, v the velocity of sound in air (at the temperature of observation), N the frequency of the note,

$$N = \frac{v}{4l}.$$

This result would, if accurate, be independent of the particular whistle employed for the measurement, but there are well known reasons for regarding the assumptions mentioned above as only

very roughly approximate in consequence of the narrowness of the pipe in which the resonance takes place.

At Mr Horace Darwin's request we undertook to attempt some measurements of the pitch of the notes sounded by such whistles by a method which would be independent of the calculation of the pitch from the pipe-length, and in this way to test the accuracy of the calculation of pitch by that method. The plan which seemed most promising was to obtain the true wave-lengths of the notes in free air by the use of a sensitive flame in the way suggested by Lord Rayleigh, "Acoustical Observations", II., *Phil. Mag.* [5] VII. p. 153*. The formula of p. 90 could then be safely employed to calculate the vibration frequency of the note.

A number of observations were taken on this plan, details of which are given below.

The wave-length measured in this way was always considerably greater than four times the length of the whistle-pipe, and it varied appreciably with the pressure of the air with which the whistle was blown: and here a difficulty as to the interpretation of the results should be mentioned. In order to obtain the nodes the whistle must be blown continuously; this was done by means of a heavily loaded gas-bag, the pressure being measured in the usual way by a U-tube; but when the whistle is used for testing the limit of audibility it is blown by the sudden puffs at an uncertain and doubtless varying pressure, obtained by compressing the rubber bladder. For a particular length of pipe one particular pressure gives the clearest note, and it is possible that this is the note which the ear regards as the note sounded when the bladder is squeezed, and that the different lengths of whistle should be compared at those particular pressures. This however is matter of speculation rather than of experimental evidence.

An interesting point is suggested by the occurrence of nodes for very short wave-lengths. The flame could be made to flare by the whistle however short the length of the pipe might be, but the shortest wave-length for which we have been able to obtain nodes is 15·84 mm. corresponding to a vibration frequency of 21,517 complete vibrations per second. This value was obtained with the reading 3 mm. of the length of the whistle-pipe, and with the same whistle the extreme limit of audibility for our own ears was as follows

F. M. T. shortest length of whistle-pipe 3·8 mm.
W. N. S. .... 3·7 mm.

so that distinct nodes were obtained when the sound was inaudible. It is however very difficult to get them satisfactorily and no good

* Reproduced in Glazebrook and Shaw's *Practical Physics*, p. 180.

result could be got with wave-lengths less than 15 mm. This may be due to the fact that the area of the section of the flame becomes then too large in proportion to the wave-length of the sound for the node to be identified, moreover the flame is then very sensitive, and there may be external disturbances which produce a continuous flaring if the flame is made sufficiently sensitive to respond to the action of the whistle, and which cannot be otherwise appreciated, or the particular flame used may be unsuitable for such very high notes. These suggestions require investigation before the fourth alternative is adopted, viz. that the action of the whistle is not continuous beyond the wave-length corresponding to about 3 mm. length of pipe.

The notes emitted by three whistles (*A*, *B*, *C*) were tested by the method described by Lord Rayleigh. The pressure of the air blowing the whistle and the length of the whistle were both capable of being varied. The pressure of the gas supplying the sensitive flame could also be varied.

Whether the flame flares or not in any given position depends not only on whether it is at a node or internode, but also on the state of the flame itself. If it is supplied by gas at too low a pressure it will not flare to any sounds, if at too high a pressure it flares spontaneously. If the pressure is raised from a low value upwards the flame first becomes sensitive to notes of wave-length about 15 mm. For notes of long wave-length (up to about 36 mm.) the flame has to be on the point of flaring, or even flaring slightly by itself before it becomes sensitive. Not only does the flame vary in sensitiveness absolutely, but also as to the difference in its behaviour at nodes and loops, which greatly affects the experiments. In some cases it is possible by adjusting the pressure to get the flame to flare well in the loops and to keep steady or with only spasmodic flaring at the nodes. The range of the observations was limited by the fact that these favourable conditions could not be obtained at either extreme. In observing the two shortest wave-lengths (16 and 16·4 mm.) the flame flared in all positions, but at the nodes the flaring was slightly but definitely less. With shorter wave-lengths no nodes were distinguishable. Whether the whistle gave no definite note, or the fault lay with the flame is undecided. With notes of longer wave-length than 36 mm. the flame could be made to flare but not to shew distinct nodes.

Experiments were also arranged to test (1) whether the formula, wave-length = 4 × length of pipe, holds good; (2) whether the different whistles give the same note; (3) whether the note varies with the pressure of air blowing it. Instead of giving all the measurements one series is quoted as a sample, and a table of all the results placed after it.

Whistle A.

Length of whistle 7·1 mm.

Pressure of air blowing the whistle = 26·4 cm. of water.

No. of the node.	Observations of distance from the plate in mm.					Mean distance.	Calculated half wave-length.
1	18·2	18·4	18·2			18·3	18·3
2	33·1	34·0	32·7	34·8		33·6	16·8
3	48·0	50·0	51·7	49·0		49·7	16·6
4	68·5	66·0	68·0	64·5	65·1	66·4	16·6
5	83·2	81·0	82·2	83·8		82·5	16·5
6	101·0	99·3				100·2	16·7
7	116·3	119·4	118·0			117·9	16·6

In this and some other cases the length calculated from the first node is far greater than that from any of the others. This may be due to the heating effect of the flame which cannot well be allowed for. We thought better therefore to leave out the first position in all cases. The mean result of this series then will be

Half wave-length = 16·63 mm.

Table of Results.

Whistle used.	Pressure of air in cm. of water.	Length of whistle in mm.	Quarter wave-length in free air in mm.
A	16·7	3·0	3·96
"	16·9	3·3	4·15
"	27·5	5·0	6·65
"	26·0	"	6·53
"	19·3	"	7·40
B	32·8	"	6·30
"	21·4	"	7·05
C	29·1	"	6·32
"	22·8	"	6·50
A	26·4	7·1	8·32
B	24·5	"	8·22
"	15·9	"	8·77
A	27·0	7·6	8·55

From these results we may conclude (1) that the wave-length in free air is considerably greater than four times the length of the whistle. (2) They do not shew any marked difference between the whistles. (3) The wave-length perceptibly diminishes, i.e. the pitch increases, as the pressure of air increases.

As to the second result, although the figures do not shew distinctly any difference between the whistles, a difference is easily heard by the ear at certain wave-lengths. If they are all supplied from the same air-bag and set to the same length, say 5 mm., one will give a good clear note while another only gives a fizz. Owing to this difference the measurements could not be taken with different whistles at the same pressure. The flame however shewed the existence of definite nodes in many cases when the sound seemed by hearing very impure, the note being well within the limits of audibility.

The difference between the wave-length in free air and the quadruple of the length of the whistle may be attributed partly to the diminution of velocity of the sound in the narrow pipe, but mainly to the fact recognised, even in the case of pipes of ordinary diameter, that the length between the whistle opening and the closed end requires a correction so that the formula for the frequency is $N_1 = \frac{v}{4(l+x)}$ where x is the correction to the measured length.

For a brass pipe, 40 millimetres in diameter and length 7 times as great, Wertheim found the value of x to be 1·5 times the diameter. Assuming the correction proportional to the linear dimensions, the correction in the case of the whistles would be about 1·1 mm. This correction would for particular pressures make the calculated pitch agree with the observed pitch for the lengths 5·0 mm., 7·1 mm. and 7·6 mm., though the correction would be too large for the observations at 3·0 mm. and 3·3 mm.

We have not been able to find sufficient data to enable us to calculate the effect of the pipe upon the velocity of sound. The Helmholtz-Kirchhoff formula, $c = C\left(1 - \frac{\gamma}{2r\sqrt{\pi n}}\right)$, quoted by Wüllner (Vol. I. p. 799), has been investigated experimentally by Schneebele and Ad. Seebeck and verified in certain cases, but doubt was thrown upon the calculation of the correction by substitution of the value of n for a note of given pitch. As the pitch of the notes we have to deal with is very high indeed, this objection becomes very important. In fact if as suggested $\frac{1}{n}$ should be taken instead of $\frac{1}{\sqrt{n}}$ the correction for our results would be thereby reduced from 2 p.c. to 1/20th per cent.

(3) *On a Class of Spherical Harmonics of Complex Degree with application to Physical Problems.* By E. W. HOBSON, M.A.

(*Abstract.*)

This paper presents an investigation of the properties of harmonics of degree, $-\frac{1}{2}+p\sqrt{-1}$, and their application to the discussion of potential problems when the bounding surfaces are cones and spheres. In particular, the free electrification of two spherical conductors connected by a short wire of finite thickness is discussed.

This paper is being printed in full in the Transactions of the Society.

May 2, 1887.

MR TROTTER, PRESIDENT, IN THE CHAIR.

Mr F. Galton was elected a Fellow.

The following communications were made:—

(1) *On the application of Lagrange's equations to the motion of a perforated solid through a liquid when there is circulation.* By A. B. BASSET, M.A.

(2) *On the deduction of the General Dynamical Equations from the principle of Energy.* By J. LARMOR, M.A.

May 16, 1887.

MR TROTTER, PRESIDENT, IN THE CHAIR.

The following communications were made:—

(1) *On a larva of Balanoglossus.* By W. F. R. WELDON, M.A.

May 30, 1887.

MR TROTTER, PRESIDENT, IN THE CHAIR.

The following Communications were made:

(1) *On the expansions of the Theta functions in ascending powers of the argument.* By J. W. L. GLAISHER, M.A., F.R.S.

The author laid before the Society certain symbolic formulæ which he had obtained for the coefficients in the general terms in the expansions of the Theta functions, and for the Theta functions themselves.

Put $$\rho = \frac{2K}{\pi} \quad \text{and} \quad u = \frac{2Kx}{\pi} = \rho x;$$

also let $\Theta_1(u)$, $\Theta_2(u)$, $\Theta_3(u)$, $\Theta(u)$ denote the four functions

$$H(u), \quad H(u+K), \quad \Theta(u+K), \quad \Theta(u),$$

and

$$\mathrm{s}(u) = \frac{\Theta_1(u)}{\Theta'_1(0)}, \quad \mathrm{c}(u) = \frac{\Theta_2(u)}{\Theta_2(0)}, \quad \mathrm{d}(u) = \frac{\Theta_3(u)}{\Theta_3(0)}, \quad \mathrm{n}(u) = \frac{\Theta(u)}{\Theta(0)}.$$

Suppose $\mathrm{s}(u)$, $\mathrm{c}(u)$, &c. developed in ascending powers of x, and let S_n, C_n, D_n, N_n be the coefficients of the nth terms in these expansions, so that

$$\frac{\mathrm{s}(u)}{\rho} = \Sigma_0^\infty (-)^n S_n \frac{x^{2n+1}}{(2n+1)!},$$

$$\mathrm{c}(u) = \Sigma_0^\infty (-)^n C_n \frac{x^{2n}}{(2n)!},$$

$$\mathrm{d}(u) = \Sigma_0^\infty (-)^n D_n \frac{x^{2n}}{(2n)!},$$

$$\mathrm{n}(u) = \Sigma_0^\infty (-)^n N_n \frac{x^{2n}}{(2n)!}.$$

The symbolic expressions found for S_n, C_n, &c. were as follows:

$$S_n = 2^n k^{-\frac{1}{2}} k'^{-\frac{1}{2}} \rho^{-\frac{3}{2}} Q^n k^{\frac{1}{2}} k'^{\frac{1}{2}} \rho^{\frac{3}{2}}$$

$$= 4^n \left(k^{\frac{3}{2}} k'^{\frac{3}{2}} \rho^{\frac{1}{2}} \frac{d}{dh} k^{\frac{1}{2}} k'^{\frac{1}{2}} \rho^{\frac{3}{2}} \right)^n$$

$$= (2Q + R_i + R_g + R_e)^n;$$

$$C_n = 2^n k^{-\frac{1}{2}} \rho^{-\frac{1}{2}} Q^n k^{\frac{1}{2}} \rho^{\frac{1}{2}},$$
$$= 4^n \left(k^{\frac{3}{2}} k'^2 \rho^{\frac{3}{2}} \frac{d}{dh} k^{\frac{1}{2}} \rho^{\frac{1}{2}}\right)^n,$$
$$= (2Q + R_e)^n;$$
$$D_n = 2^n \rho^{-\frac{1}{2}} Q^n \rho^{\frac{1}{2}}$$
$$= 4^n \left(k^2 k'^2 \rho^{\frac{3}{2}} \frac{d}{dh} \rho^{\frac{1}{2}}\right)^n$$
$$= (2Q + R_g)^n;$$
$$N_n = 2^n k'^{-\frac{1}{2}} \rho^{-\frac{1}{2}} Q^n k'^{\frac{1}{2}} \rho^{\frac{1}{2}}$$
$$= 4^n \left(k^2 k'^{\frac{3}{2}} \rho^{\frac{3}{2}} \frac{d}{dh} k'^{\frac{1}{2}} \rho^{\frac{1}{2}}\right)^n$$
$$= (2Q + R_i)^n;$$

where
$$h = k^2, \quad h' = k'^2,$$
$$R_i = \frac{4KI}{\pi^2}, \quad R_g = \frac{4KG}{\pi^2}, \quad R_e = \frac{4KE}{\pi^2},$$

and Q is an operator defined by the equation *

$$Q\left(R_i^{\alpha} R_g^{\beta} R_e^{\gamma}\right) = (\beta+\gamma) R_i^{\alpha+1} R_g^{\beta} R_e^{\gamma} + (\gamma+\alpha) R_i^{\alpha} R_g^{\beta+1} R_e^{\gamma}$$
$$+ (\alpha+\beta) R_i^{\alpha} R_g^{\beta} R_e^{\gamma+1} - \alpha R_i^{\alpha-1} R_g^{\beta+1} R_e^{\gamma+1} - \beta R_i^{\alpha+1} R_g^{\beta-1} R_e^{\gamma+1}$$
$$- \gamma R_i^{\alpha+1} R_g^{\beta+1} R_e^{\gamma-1}.$$

The following symbolic forms for the Theta functions are deducible from the above values of the coefficients :

$$\mathrm{s}'(u) = h^{-\frac{1}{4}} h'^{-\frac{1}{4}} \rho^{-\frac{3}{2}} \cos\left(2x h^{\frac{1}{2}} h'^{\frac{1}{2}} \rho \frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\right) h^{\frac{1}{4}} h'^{\frac{1}{4}} \rho^{\frac{3}{2}},$$

$$\mathrm{c}(u) = \quad h^{-\frac{1}{4}} \rho^{-\frac{1}{2}} \cos\left(2x h^{\frac{1}{2}} h'^{\frac{1}{2}} \rho \frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\right) h^{\frac{1}{4}} \rho^{\frac{1}{2}},$$

$$\mathrm{d}(u) = \quad \rho^{-\frac{1}{2}} \cos\left(2x h^{\frac{1}{2}} h'^{\frac{1}{2}} \rho \frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\right) \rho^{\frac{1}{2}},$$

$$\mathrm{n}(u) = \quad h'^{-\frac{1}{4}} \rho^{-\frac{1}{2}} \cos\left(2x h^{\frac{1}{2}} h'^{\frac{1}{2}} \rho \frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\right) h'^{\frac{1}{4}} \rho^{\frac{1}{2}};$$

$$\mathrm{s}'(u) = \cos\left(2 h^{\frac{3}{8}} h'^{\frac{3}{8}} \rho^{\frac{1}{4}} \frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}} h^{\frac{1}{8}} h'^{\frac{1}{8}} \rho^{\frac{3}{4}} x\right),$$

* The operator $Q = 2q \frac{d}{dq} = 2hh'\rho^2 \frac{d}{dh}$. The quantities I, G, E have been considered in Vol. v. pp. 184—208, 232—250.

$$\mathrm{c}\,(u) = \cos\left(2h^{\frac{3}{8}}\,h'^{\frac{1}{2}}\,\rho^{\frac{3}{4}}\,\frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\,h^{\frac{1}{8}}\,\rho^{\frac{1}{4}}\,x\right),$$

$$\mathrm{d}\,(u) = \cos\left(2h^{\frac{1}{2}}\,h'^{\frac{1}{2}}\,\rho^{\frac{3}{4}}\,\frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\,\rho^{\frac{1}{4}}\,x\right),$$

$$\mathrm{n}\,(u) = \cos\left(2h^{\frac{1}{2}}\,h'^{\frac{3}{8}}\,\rho^{\frac{3}{4}}\,\frac{d^{\frac{1}{2}}}{dh^{\frac{1}{2}}}\,h'^{\frac{1}{8}}\,\rho^{\frac{1}{4}}\,x\right);$$

and

$$\mathrm{s}'\,(u) = \cos\left\{\left(R_i + R_g + R_e + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}} x\right\},$$

$$\mathrm{c}\,(u) = \cos\left\{\left(R_e + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}} x\right\},$$

$$\mathrm{d}\,(u) = \cos\left\{\left(R_g + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}} x\right\},$$

$$\mathrm{n}\,(u) = \cos\left\{\left(R_i + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}} x\right\}.$$

In the last group R denotes ρ^2, so that

$$R = \frac{4K^2}{\pi^2}.$$

In $\mathrm{s}'(u)$ the accent denotes differentiation with respect to u; the expressions for $\mathrm{s}'(u)$ have been given in preference to those for $\mathrm{s}\,(u)$ as they are analogous in form to those of the functions $\mathrm{c}\,(u)$, $\mathrm{d}\,(u)$, $\mathrm{n}\,(u)$. The formulæ for $\mathrm{s}\,(u)$ may be at once obtained by integration with respect to x. Taking the last group, for example, we have

$$\frac{\mathrm{s}\,(u)}{\rho} = \frac{\sin\left\{\left(R_i + R_g + R_e + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}} x\right\}}{\left(R_i + R_g + R_e + 4hh'R\,\frac{d}{dh}\right)^{\frac{1}{2}}}.$$

(2) *Some Laboratory Notes.* By W. N. Shaw, M.A.

The author exhibited the following experiments:

1. *Resolving power of a telescope.*

The arrangement was a modification of that described by Lord Rayleigh (*Phil. Mag.* August 1880). Instead of a diaphragm with unalterable aperture an ordinary adjustable slit such as is used for the projection of a spectrum was mounted imme-

diately in front of the object-glass of the telescope. The slit was sufficiently long to extend right across the object-glass, and was set vertical. At a fixed distance from the telescope a piece of wire-gauze of fine mesh was set up with its wires horizontal and vertical; behind it was a ground-glass plate illuminated by a sodium flame. The telescope was then focussed on the wire-gauze, the slit being set wide open. If the distance was not too great a well-defined image of the gauze was obtained. The size of the slit was then gradually reduced by turning its screw until it became so narrow that there was not sufficient aperture for the resolution of vertical lines (those parallel to the slit). The appearance then

Mesh of gauze.	Distance of gauze from O.-G. of telescope.	Consecutive readings of width of slit by wedge for positions of disappearance of the vertical lines.	Mean width reduced to inches.	Calculated wave length in tenth-metres.
31·5 wires to 1 in.	76 in.	250, 249, 250, 247, 256, 247	·054	5730
,,	117 ,,	370, 362, 365, 370, 368, 355, 365	·085	5858
,,	151·5 ,,	460, 468, 475, 470	·112	5960
60 to 1 in.	76·5 ,,[1]	438, 445, 437, 450 430, 450	·102	5640
,,	116·5 ,,	680, 695, 700, 685	·164	5959

presented in the telescope was a field covered by horizontal wires only. The slit was adjusted to the position at which the vertical wires were just no longer resolved, and its width was then measured by means of a graduated wedge. The position proved to be fairly definite and easily recognisable. The slit must be narrowed until all trace of vertical lines is gone; the tendency seemed to be to judge them as disappeared before the slit was sufficiently narrow. To verify Lord Rayleigh's theory with regard to the resolving power, the measurements required are as described in the paper mentioned, viz. l the distance of the gauze from the object-glass, d the distance between corresponding parts of consecutive wires of the gauze, and Δ the width of the slit; from these the

[1] At this distance the want of uniformity in the mesh of the gauze was clearly apparent, and made observations uncertain.

wave length of the light can be calculated. The Table gives some results obtained in the laboratory for the wave length of sodium light.

The accuracy is of course not of a very high order, but it is remarkable that so close an approximation can be got with apparatus that is in common use in all laboratories.

2. *An arrangement to measure the length of an object against which a scale cannot be laid.*

This method was devised to measure the wave lengths of standing waves on the surface of a mercury trough; it is applicable in many other cases. The object to be measured is viewed by a telescope, and a piece of plane parallel glass is interposed obliquely between the object and the telescope, and so adjusted as to reflect into the telescope the light from a scale. The scale must be placed so that its reflected image in the plane glass coincides with the surface of mercury or other object whose length is to be measured. The length can be easily read off in this way if proper care be taken in adjusting the relative illumination of the scale and the surface. An ivory scale with a black background answers very well for the purpose. The plate of glass should be thick, as two images of the scale are formed, and they will confuse one another unless they are separated by a considerable thickness of glass. The method is of course similar in principle to the use of the Wollaston *camera-lucida* and to the scale reading of a spectrum by reflection from the face of the prism.

3. *A lecture experiment in self-induction.*

It is known that if the poles of an electro-magnet are short circuited by an incandescent lamp, the lamp may be made to glow for an instant on breaking the battery circuit, although the E.M.F. may not be sufficient to make it glow when the current through the circuit is steady. This experiment may be made more striking by interposing in the battery circuit a revolving wheel contact breaker instead of a key. By means of an open wire resistance, interposed between the battery terminal and the electro-magnet, the electrode of the lamp can be attached at such a point of the battery circuit that the shunt current is just not sufficient to make it glow. On rapidly turning the wheel contact breaker the energy of the battery is drawn upon to form the magnetic field of the magnet, and discharged through the lamp, which glows well and steadily in consequence, if the contact is good and sufficiently rapidly broken.

The lamp used in illustration was a 20-volt lamp, and the battery referred to, a set of eight storage cells.

Corrigendum of F. Y. Edgeworth's paper on *Observations and Statistics* in the Cambridge Philosophical Transactions for 1885, Vol. XIV. Part II. p. 140.

Mr Edgeworth desires to retract or retouch some passages in this paper. At p. 159 in the paragraph headed (ϵ) he would cancel the statement that "in the general case" (of facility-curves differing from the probability-curve and from each other) the method of least squares is not "theoretically correct". The method is defensible in that case on the same principle as that on which it is defended in the less general case headed (δ) (p. 158) (when the facility-curves appear in homogeneous clusters). The principle is, that it is allowable to ignore (especially if we are ignorant of) part of the data (or danda), namely the specialities of the facility-curves; and to utilise only our knowledge (1) of the Arithmetical Mean (simple or weighted) of the observations, and (2) of the inverse-mean-square-of-error for the respective facility-curves. *Confining our attention to these two circumstances,* we may reason by Inverse Probability that the probability of the observed Arithmetic Mean differing from the real point by any assigned extent x is measured by (the integral of) a Probability-curve $y = \frac{1}{\sqrt{\pi}c} e^{-\frac{x^2}{c^2}}$, whose modulus-squared c^2 is the mean (simple or weighted) of the inverse-mean-square-of-error for the different facility-curves. It cannot be denied that there is something arbitrary in selecting one portion of our information to be utilised (for instance one weighted Arithmetic Mean, rather than another). The practice is explained by the parallel procedure of an Insurance Company, when they cannot utilise all their information. As pointed out by Dr Venn in his *Logic of Chance,* (chap. VIII. § 22 et seq.) they have often a choice as to what portion of the data is to be neglected. For example given mortality statistics for (α) English-residents-in-Madeira, and (β) Consumptive-residents-in-Madeira; upon which basis should an English-consumptive-resident be insured? Either plan would come right in the long run; but the run would be much longer, the deviations between the average and the particulars would doubtless be much greater, upon plan (α); since "the state of a man's lungs has probably more to do with his health than the place of his birth has". Similarly any weighted Arithmetic Mean will come right in the long run of numerous applications of the method. But some systems of weight will afford much shorter excursions from the true Mean, will deviate less from the real point in the course of repeated use, than others. The method of least squares assiges the system which (as compared with other linear combinations of the observations) is attended with the

minimum deviation. The same principle on which we prefer one species of Arithmetical Mean to another is required when we choose between the Arithmetical Mean and other genera. The comparison between the Arithmetical Mean and the Median made at p. 168 of the paper referred to presupposes this principle. These considerations produce some softening of the strictures on the process described as "assumed inversion" (p. 153 et seq.). The writer has attempted to treat the subject with more clearness than is admitted by the brevity of this note in a study on the Art of Measurement, entitled *Metretike* (London, Temple Co. 1887).

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 31, 1887.

ANNUAL GENERAL MEETING.

PROFESSOR ADAMS, VICE-PRESIDENT, IN THE CHAIR.

THE following Officers and new Members of Council were elected:—

President:

Mr Trotter.

Vice-Presidents:

Prof. Foster, Prof. Stokes, Prof. Cayley.

Treasurer:

Mr J. W. Clark.

Secretaries:

Mr Glazebrook, Dr Vines, Mr Larmor.

New Members of Council:

Prof. Adams, Dr Routh, Dr Lea, Mr Harmer.

Prof. Stokes, Vice-President, then took the Chair and returned thanks for the President and other Officers.

The names of the Benefactors of the Society were read by the Secretary.

Mr S. J. Hickson, M.A., Downing College, was elected a Fellow.

Prof. Newton presented to the Society on behalf of Mr Leonard Blomefield (formerly Jenyns), a small work by himself entitled Chapters in my Life, and a vote of thanks to the Author was proposed and carried unanimously.

The following communications were made:

(1) *On a special algebraic function and its application to the solution of some equations.* By Sir G. B. Airy, K.C.B., M.A., LL.D.

In studying the elegant resolution, by Professor Adams, of the expression $x^n + \frac{1}{x^n} - 2\cos n\alpha$ into Factors, it has occurred to me that an equivalent solution may be given, by a somewhat different process, of which the first step has not been usually recognized, but which may possibly be useful in other cases. I propose to apply it, in the first instance, to the equation $x^n - 1 = 0$. A new function, with a new symbol, will be employed.

We shall use the symbol

$$\psi(\theta) \text{ for the expression, } \cos\theta + \sqrt{-1}\,.\sin\theta;$$

and therefore $\psi(-\theta)$ .... $\cos\theta - \sqrt{-1}\,.\sin\theta$.

Now the function ψ possesses these properties:—

$$(1)\quad \psi(\theta) \times \psi(-\theta) = 1.$$

$$(2)\quad \psi(p) \times \psi(q) = \psi(p+q).$$

$$(3)\quad \{\psi(p)\}^n = \psi(np).$$

(These properties are analogous to those which apply to the function ϵ^θ.)

We shall now consider the application of these equations to the solution of the equation $x^n - 1 = 0$, or $x^n = 1$, where n is an odd prime number: (the consideration of the prime 2 will quickly be found to be unnecessary).

The equation (3) above will evidently give

$$x = \cos p + \sqrt{-1}\,.\sin p$$

for the solution required, provided that we make

$$\psi(np) = 1,$$

or

$$\cos(np) + \sqrt{-1}\,.\sin(np) = 1.$$

This implies two conditions. First, np must be some angle which produces $\sin(np) = 0$, that is, $np = 0$, or $= \pi$, or $= 2\pi$, or $= 3\pi$, &c. Secondly, we must preserve the condition $\cos(np) = +1$. This excludes the values π, 3π, 5π, &c.; and therefore we must retain only $np = 0$, or $= 2\pi$, or $= 4\pi$, &c. to $(2n-2)\pi$; after

which, the terms $2n\pi$, $(2n+2)\pi$, &c., give the same values for sine and cosine which are given by 0, 2π, 4π, &c. Thus we obtain, as available for p, the values $0, \frac{2\pi}{n}, \frac{4\pi}{n}$, &c., as far as $\frac{(2n-2)\pi}{n}$. And we have, for values of x,

$$\cos(0),$$

$$\cos\frac{2\pi}{n} + \sqrt{-1}\,.\sin\frac{2\pi}{n},$$

$$\cos\frac{4\pi}{n} + \sqrt{-1}\,.\sin\frac{4\pi}{n},$$

&c., as far as

$$\cos\frac{(2n-2)\pi}{n} + \sqrt{-1}\,.\sin\frac{(2n-2)\pi}{n}.$$

But the condition $\sin(np) = 0$ is satisfied equally well by negative values of 2π, 4π, &c.; which will not alter the terms $\cos\frac{2\pi}{n}$, $\cos\frac{4\pi}{n}$, &c., but will change the signs of $\sin\frac{2\pi}{n}$, $\sin\frac{4\pi}{n}$, &c., and thus give for values of x,

$$\cos\frac{2\pi}{n} - \sqrt{-1}\,.\sin\frac{2\pi}{n},$$

$$\cos\frac{4\pi}{n} - \sqrt{-1}\,.\sin\frac{4\pi}{n},$$

&c., as far as

$$\cos\frac{(2n-2)\pi}{n} - \sqrt{-1}\,.\sin\frac{(2n-2)\pi}{n}.$$

In order to form the series of terms whose continued product will be equal to the given quantity $x^n - 1$, we must connect, negatively, each of these values of x with the symbol x, and multiply all together. The multiplicands, thus formed, are

$$x - 1,$$

$$x - \left(\cos\frac{2\pi}{n} + \sqrt{-1}\,.\sin\frac{2\pi}{n}\right),$$

$$x - \left(\cos\frac{4\pi}{n} + \sqrt{-1}\,.\sin\frac{4\pi}{n}\right),$$

&c., in one series; and

$$x - \left(\cos\frac{2\pi}{n} - \sqrt{-1}\,.\sin\frac{2\pi}{n}\right),$$

$$x - \left(\cos\frac{4\pi}{n} - \sqrt{-1}\,.\sin\frac{4\pi}{n}\right),$$

&c., in the other series.

Uniting, by multiplication, the homologous terms from the two series, after $x-1$, and referring to equation (1) above, we find for the terms required,

$$x - 1,$$

$$x^2 - 2x\,.\cos\frac{2\pi}{n} + 1,$$

$$x^2 - 2x\,.\cos\frac{4\pi}{n} + 1,$$

&c.,

$$x^2 - 2x\,.\cos\frac{(2n-2)\,\pi}{n} + 1,$$

whose continued product will form $x^n - 1$.

We will now apply the same principles to Professor Adams' Formula, or $\left(x^m + \dfrac{1}{x^m}\right) - 2\cos n\alpha.$

I premise that

$$2\cos n\alpha = (\cos n\alpha + \sqrt{-1}\,.\sin \alpha) + (\cos n\alpha - \sqrt{-1}\,.\sin n\alpha)$$
$$= \psi\,(n\alpha) + \psi\,(-n\alpha);$$

and the Formula becomes

$$x^m + x^{-m} - \psi\,(n\alpha) - \psi\,(-n\alpha),$$

or
$$x^m - \psi\,(n\alpha) + x^{-m} - \psi\,(-n\alpha).$$

This expression at once suggests that, to make the Formula $=0$, x^m must $=\psi\,(n\alpha)$, and x^{-m} must $=\psi\,(-n\alpha)$; (which two relations, by virtue of Equation (1), are equivalent). And, if

$$x^m = \cos m + \sqrt{-1}\,.\sin m + \psi\,(m),$$

the Formula becomes

$$\{\psi\,(m) - \psi\,(n\alpha)\} + \{\psi\,(-m) - \psi\,(-n\alpha)\},$$

which is satisfied by the equation

$$\psi(m) = \psi(n\alpha),$$

or
$$\cos(m) = \cos(n\alpha) + \sqrt{-1}\,.\sin(n\alpha)$$

an equation which can hold in every case where $\sin(n\alpha) = 0$, or $n\alpha = 0$, or $= \pi$, or $= 2\pi$, or $= 3\pi$, &c.: but which, as regards the concurrent value of m, holds only when $n\alpha = 0$, or $= 2\pi$, or $= 4\pi$, &c.; or when $\frac{m}{n} = \alpha$, or $\frac{m}{n} = \alpha + \frac{2\pi}{n}$, or $\frac{m}{n} = \alpha + \frac{4\pi}{n}$, &c.

The remaining steps are the same as those in Professor Adams' Memoir.

(2) *Some observations on Permanganic Acid.* By T. H. EASTERFIELD.

WHEN potassium permanganate is dissolved in strong sulphuric acid, a green liquid is obtained, which, as is well known, is possessed of considerable oxidising power, and is decomposed with explosion when the temperature is slightly raised. Terreil, by using monohydrated sulphuric acid, $H_2SO_4 . H_2O_1$, and distilling the liquid so obtained upon the water bath, at a temperature of 50°—70°, succeeded in obtaining purple vapours; these vapours condensed in the receiver to a dark crystalline mass, soluble in water, violently decomposed by a sudden elevation of temperature, and supposed by Terreil to be pure permanganic acid, $HMnO_4$ or $H_2Mn_2O_8$. The solution of this body in water is of the same colour as a solution of potassium permanganate, and some chemists have supposed that the crystalline distillate was permanganic anhydride Mn_2O_7, the solution being an aqueous solution of permanganic acid. As far as we are aware, however, no analyses of the crystalline distillate have as yet been published; partly no doubt from the danger attending its preparation.

The primary object of the following experiments was to determine the density of the purple vapour above mentioned; but though this object was not attained, the facts which came under notice appeared of sufficient interest for publication.

It was suggested by Mr H. J. H. Fenton (to whose aid throughout my experiments I am greatly indebted, and from whom came the proposition of the Vapour Density determination), that the anhydrides of phosphoric or other acids might liberate permanganic anhydride from potassic permanganate. The following experiments were accordingly made:—

(i) Well powdered potassium permanganate was dried for several hours at 120° in an air bath. A very slight amount of

decomposition took place, oxide of manganese being probably produced. This permanganate was then mixed with about half its weight of phosphoric pentoxide, and the mixture gradually heated. No change appeared to take place at the temperature of 100°, but when the mixture was still further heated over a small gas-flame, action suddenly began at one point, and rapidly spread throughout the whole mass, though the lamp was at once removed.

So long as the action was going on, pink vapours were given off, which condensed to small solid particles in the neck of the retort in which they were generated. These particles, however, gradually liquefied to a solution of the characteristic permanganate colour. This liquefaction was probably due to the fact that the phosphorus pentoxide used contained a certain quantity of water, which appears to be driven off when the pentoxide is acted upon by the permanganate and to dissolve the solid permanganic acid, after condensing in the neck of the retort. As a further proof of this, drops of colourless water condensed on certain parts of the neck of the retort. The products of distillation were tested for potassium and phosphoric acid, but gave no trace of either. The pink liquid was faintly acid to litmus paper. The quantity of distillate obtained was very small compared with the quantities of phosphoric pentoxide and permanganate taken. The yield was slightly increased by mixing the ingredients with fine sand so as to prevent the too rapid spreading of the reaction through the mass. When the reaction had once begun, and had been allowed to run to a finish, the application of heat caused no further evolution of the pink gas, though a little more water was expelled. It was noticed that if the mixture of permanganate and pentoxide was allowed to absorb moisture, by exposure to the atmosphere of the room for a few hours, the yield of pink vapours was considerably increased. When large excess of pentoxide was used a considerable amount of it volatilised unchanged and only a trace of the solid permanganic acid was obtained. (No attempt was made to ascertain the composition of the fused mass which remained in the retort.)

(ii) Dry potassium permanganate was ground up with boric anhydride and the mixture heated. The phenomena observed were almost identical with those noticed when phosphoric pentoxide was used, so long as the boric anhydride contained a small quantity of moisture. The smaller the quantity of moisture present, the smaller the quantity of pink vapour and liquid produced. A little boric acid came off with the vapours, volatilising in the steam given off. In order to prevent any possibility of spirting in these experiments, the products of distillation were made to pass through a thick layer of glass wool. When a drop or two of

water was added to a small quantity of the finely ground mixture of permanganate and boric anhydride a considerable evolution of heat occurred, and a comparatively large quantity of pink liquid condensed in the upper part of the test tube. The fused mixture of boric anhydride and potassic permanganate was a glassy mass of a fine purple colour, insoluble in water, or at any rate communicating no colour to water when boiled with it for a few minutes. It was but slowly decomposed by aqua regia in the cold, but with greater rapidity upon boiling. It appeared to impart its colour to the glass of the tube in which it was formed.

(iii) Potassium permanganate was also heated with anhydrous copper sulphate, and with potassic bichromate, but in neither case were results of a definite character obtained.

Assuming that the pink vapours obtained in these experiments were identical with those obtained when permanganate is heated with slightly diluted sulphuric acid, the evidence obtained seems to point to the fact that these vapours are permanganic acid rather than permanganic anhydride, since no appreciable amount of coloured vapour is produced unless some water be present in each case.

Though the methods above described for the preparation of permanganic acid have the advantage over that of Terreil in being unattended with danger, the yield by them is so small that it was deemed advisable in the subsequent experiments to employ the old method. Though several violent explosions occurred, they were not accompanied by serious results.

The first experiment made in this manner had as its object to determine, if possible, the composition of the crystalline distillate obtained when permanganate is gently heated with monohydrated sulphuric acid.

For this purpose a small retort, with a long, gently-tapering neck, was taken, and three test tubes, which slid easily over the neck, were chosen, and fitted with small corks, marked A, B, and C, that they might be readily distinguished. A small lip was made on one side of the edge of each test tube, so that when the retort neck reached nearly to the bottom of the tube a channel should be left between the neck of the retort and the side of the test tube. The acid used was pure redistilled sulphuric, diluted with the requisite quantity of distilled water. The potassium permanganate was in all cases dried at 100° C. before being used.

The operations were conducted as follows:—The retort, containing a few grams of permanganate, was held in a clamp, so that the body dipped to the depth of two or three cm. into cold water, contained in a hemispherical bath of copper. One of the test tubes, whose weight, together with that of its cork, had been accurately determined, was deprived of its cork, and slipped over

the neck of the retort, and then surrounded by a freezing mixture of ice and salt. The monohydrated sulphuric acid was now poured into the retort through a small funnel, and the tubulus closed by a well-fitting cork.

In order that the gases might be removed from the body of the retort as fast as they were formed, a slow current of air, dried by calcium chloride and sulphuric acid, was driven through a fine glass tube passing through the cork, and dipping beneath the surface of the liquid in the retort. The temperature of the water in the bath was now gradually raised by a small Bunsen lamp. When the temperature had risen to 40°, pink vapours filled the retort, and were condensed in the test tube to a mass of radiating crystals of a deep red colour. After about half an hour, the test tube was removed and another put in its place. The test tube which had been removed was wiped round its edge with a clean dry cloth, so as to remove any products of condensation which might attack the cork. The tube was then corked, and, after being wiped free from the water which adhered to its outer surface, was placed in the balance-case for ten minutes and then weighed. The increase in weight gave the weight of the substance condensed in the tube. The tube was now carefully washed out with distilled water. The entire contents of the tube were thus removed as a pink solution, with the exception of a thin brown film (probably oxide of manganese) adhering firmly to the tube. The tube was now dried and again weighed, but only in one case could an increase of weight, due to this film, be detected.

To the pink solution obtained by rinsing out the test tube with distilled water, 10 cc. of dilute standard oxalic acid solution were added, together with a few drops of dilute sulphuric acid. The liquid was thus completely decolourized. It was now warmed in the water bath to about 60° and titrated back with approximately centinormal permanganate. The difference between the volume of centinormal permanganate required to oxidise 10 cc. of the standard oxalic acid and the volume used in titrating back the oxalic acid, to which the contents of the test tube had been added, gave the volume of centinormal permanganate equivalent in oxidising power to the contents of the test tube. From this the oxidising power, or available oxygen in the distillate, was easily calculated. Since the weight of the distillate was known the percentage of available oxygen could be found.

The results were as follows:—

Tube.	Weight of distillate.	Equivalent volume of centinormal permanganate.	Percentage of available oxygen.
A	·0055 gm.	6·4 cc.	9·07
B	·0067 „	9· „	10·75
C	·0065 „	9·4 „	11·28

The difference between these numbers is considerable and can hardly be attributed to errors of experiment, but rather to the fact that the distillate obtained in the experiments was not constant in composition.

The percentage of available oxygen in permanganic acid, $H_2Mn_2O_8$, would be 33·33, or more than three times as great as the mean amount found by experiment. Hence the substance could not be pure permanganic acid.

It is however possible that the distillate is a mixture of permanganic acid with a quantity of water insufficient to liquefy it, aqueous vapour having been removed by the current of air from the diluted sulphuric acid in the retort. On this supposition the mixture would contain about 30 °/$_{\circ}$ of permanganic acid.

As the product obtained by distillation of permanganate with slightly diluted sulphuric acid could not be regarded as a body of constant composition, a determination of the density of this product in the gaseous state appeared likely to be of little value.

Acting upon the suggestion above made that the distillate was a mixture of permanganic acid and water, it was resolved to try the effect of pure sulphuric acid in the place of the monohydrated acid used in the preceding experiments. Accordingly a few drops of Nordhausen acid were poured upon a small quantity of potassium-permanganate in a test tube surrounded by ice, but the reaction was of such a violent nature that no further experiments were attempted with this reagent.

In the next experiment pure re-distilled sulphuric acid was poured on to potassium-permanganate contained in a retort resting in a bath of cold water. The tubulus was closed by a cork, with a glass tube arranged as in the experiment with the diluted acid, so that dry air might be passed through the liquid in the retort if necessary. A piece of caoutchouc tubing with a screw-clip allowed the tube to be opened or closed at will. The neck of the retort passed air-tight through a caoutchouc cork nearly to the bottom of a strong wide test tube. This cork was perforated for the air-tight passage of another tube, which allowed the gases produced in the reaction to escape into the atmosphere. When the pure sulphuric acid had been poured upon the permanganate in the retort, no pink vapours were given off, notwithstanding that the temperature was cautiously raised to 70°, and that bubbles of gas were being given off from the surface of the mixture. The exit tube of the apparatus was therefore connected up with a Bunsen air-pump giving a vacuum of about 450 mm. of mercury, so as to reduce the pressure inside the apparatus. The rate of evolution of the gases was at once increased, but the retort and test tube attached to it shewed no signs of pink vapour or distillate, notwithstanding that

the test tube was surrounded by ice-cold water. After the experiment had been allowed to go on for some time however, it was noticed that though the apparatus itself was perfectly colourless, the glass tubes of the water air-pump were distinctly pink, and became gradually more so as the experiment proceeded. The question naturally suggested itself, "Why should the evolved gases colour the tubes of the pump and not the neck of the retort and the test tube through which they had first passed?" The only answer to this question seemed to be that the tubes of the pump were moist, and that a gas was given off from the sulphuric acid and potassium permanganate which, though colourless by itself or in the absence of water, was dissolved by water with the formation of a pink solution. To test whether this supposition was correct, the evolved gases were now passed through a second test tube, to catch any spirtings which might possibly come over from the mixture in the retort, and then through bulbs containing distilled water, dry air being drawn through the retort, so as to remove the gases as fast as they were formed. In a few minutes the water in the bulbs became pink, and the sides of the entrance tube of the bulbs, being moist, first became pink, and afterwards dark brown in colour. When the bulbs were subsequently rinsed out, it was found that though they themselves became perfectly clear, a brown deposit was left in this entrance tube, as though the body formed was stable in the presence of a large excess of water, but decomposed by a small quantity of the same liquid. The pink solution from the bulbs was reduced with oxalic acid, and then yielded a colourless solution, containing no trace of sulphuric acid or potassium, so that the coloration could not be due to spirtings from the retort. When the yield of the gas was good it was found that though the mixed gases were colourless before entering the bulbs, they were distinctly purple as they escaped from the exit end of the apparatus.

A piece of wide glass tubing was now drawn off at each end to a thin tube, and the gases from the retort passed through it for some time. It remained perfectly colourless, but upon removing it, placing a few cc. of distilled water inside it, and shaking up, a pink liquid was obtained. This liquid was very faintly acid to litmus paper.

This experiment was repeated with a large dry flask, and gave similar results.

Another piece of thick tubing, similar to that used in the last experiment, was now taken, and after the gases had been passed through it for about half an hour, it was heated by a Bunsen lamp. A brown ring soon formed inside the tube, at each side of the point where the flame was applied, just as, in the case of decomposition of arseniuretted hydrogen, a black ring is formed.

The ring thus formed was insoluble in water, but readily soluble in strong hydrochloric acid, and was probably some oxide of manganese.

The gas was at once decomposed by contact with mercury, causing a purple film upon the surface.

An attempt was now made to obtain some clue as to the nature of this gas by passing through weighed bulbs containing water, and provided with a calcium chloride tube, and then titrating the pink liquid so obtained with oxalic acid and centi-normal permanganate. The results were extremely unsatisfactory, giving 1·6, 2·1 and 0·8 °/ₒ as the oxygen value in three experiments. Since ozone is perceptibly soluble in water, the ozone evolved from the retort may, by dissolving in the water in the bulbs, have caused the greater part of the observed increase in weight.

Attempts were made to determine the density of this gas, but failed, owing probably to two very apparent causes: (i) The volume of the gas evolved was only a minute fraction of the air passed through the retort; (ii) ozone was largely given off.

Experiments made without passing air through the retort were quite as unsatisfactory; the yield of the required gas appeared smaller, whilst that of oxygen and ozone increased.

In the vapour density experiments, the gases were first passed through a small quantity of pure redistilled sulphuric acid at the bottom of the second test tube in the apparatus already mentioned, then through a conical flask, and finally into a specially made bulb. This bulb, which was capable of holding rather more than half a litre was of thin glass, and had two narrow tubes entering it at opposite ends, for the entrance and exit of the gases. The tubes were closed by small glass plugs, kept in their places by little pieces of caoutchouc tubing. The exhaust tube was connected with a Mariotte's bottle, so that a regular current of air could be drawn through the entire apparatus when required.

When the apparatus was set going, the manganese gas was at first all absorbed in the sulphuric acid through which it passed, and the acid became olive green, the colour deepening until the acid was saturated, after which the gases passed through apparently unchanged. In no case did the increase of weight in the bulb above that of the bulb filled with dry air amount to more than two centigrammes, and since ozone was no doubt present, satisfactory results could hardly be expected.

A calcium chloride tube was placed between the aspirating bottle and the vapour density bulb, to prevent aqueous vapour diffusing back. The arrival of the gas was at once indicated by a change in colour of the calcium chloride from white to reddish brown.

The yield of the gas under consideration appears to vary

greatly with the temperature and the proportions of the ingredients employed in the retort. Rapid removal from the retort by a current of dry air appears to favour the production of the gas. Below 50° the gas does not appear to be given off, or at any rate only in minute quantities. Above 70° the gas obtained is nearly pure oxygen with very little of the manganese compound. If the volume of pure sulphuric acid employed is only about twice that of the permanganate, a violent explosion is almost certain to occur when the temperature reaches 60°. If still less acid be employed the risk of explosion appears to be increased. By increasing considerably the proportion of acid all risk of explosion may be obviated. The best yield of gas was obtained in an experiment where the proportions were so adjusted that a very slight explosion took place when the temperature reached 65° by allowing the temperature to fall to 60° and then passing air through the retort as if no explosion had occurred. The exact proportions of acid and permanganate had unfortunately not been previously determined.

When the spent liquor from the retort was dropped into water a copious evolution of the pink vapours occurred. This method was tried as a means of obtaining crystalline permanganic acid, and gave satisfactory results. The following experiments were subsequently tried with the gas:—

The vapour density bulb was removed and a large flask, well dried by the passage of dry air through it, substituted. The flask had a well fitting cork, through which passed three tubes. Two of these reached nearly to the bottom of the flask, one for the delivery of the manganese gas, the other for that of any other gas, whilst the third served as an exhaust pipe. The flask was now filled with air which had passed through the generating retort, and which carried with it the manganese gas as usual. The flask remained colourless as when full of dry air. Air saturated with aqueous vapour by passing through three wash-bottles containing warm water was now sucked into the flask. A change in colour at once became evident. An opaque brown film, insoluble in water, gradually covered the sides of the flask, except immediately beneath the delivery tube, where a purplish deposit accumulated; this was partially soluble in water, with formation of a pink solution. It has already been stated that a trace of water seems to decompose the gas, whilst an excess of water dissolves it, which probably accounts for the brown deposit.

A second flask, similar to the first, was now filled in the same way with air which had passed through the retort; then dry ammonia gas was passed in. At first no action seemed to occur, but in a few minutes a brown deposit formed on the sides similar

to that in the first flask, and this deposit increased perceptibly upon standing for some time.

Finally a piece of freshly cut metallic sodium was introduced into a third similar flask containing the manganese gas. It gradually acquired a purplish tint, totally different from the appearance which sodium assumes when exposed to the air.

The existence has thus been proved of a gaseous manganese compound, which appears to be colourless but which at once yields a coloured compound in the presence of water. I now venture to suggest that this body may be and probably is, permanganic anhydride, since the solution of this body in water gives a clear pink solution, shewing all the reactions of aqueous permanganic acid.

The reasons for regarding the pink vapours of Terreil and others as permanganic acid, have been already given.

Since the above was written (August '87) it has come under my notice that Franke has published, in the *Journal für Praktische Chimie* for July and August last, two papers upon the action of Sulphuric Acid upon Potassic Permanganate. He asserts that by this action he has obtained two somewhat volatile, highly coloured solids, to which he ascribes the formula MnO_3 and MnO_4. His evidence for the formation and reactions of these bodies however appears to be somewhat unsatisfactory.

(3) *The equations of an Isotropic Elastic Solid in Polar and Cylindrical Coordinates, their solution and application.* By C. Chree, M.A.

[Abstract.]

Starting with the expression in ordinary Cartesian Coordinates for the energy of an Isotropic Elastic Solid, the author transforms this expression into polar and cylindrical coordinates, and thence obtains the corresponding internal and surface equations.

To the development of these equations one general method is applied. Solutions are obtained for the internal equations involving arbitrary constants, which are then determined from the surface conditions.

A general solution of the polar equations is obtained in the case of a sphere or a spherical shell, for a spherical harmonic distribution of internal and surface forces. This solution is extended to the case of any number of materials forming concentric spherical shells, when the force on the bounding surface is purely normal; and it is shewn by this means how to treat a material whose structure varies continuously with the distance

from some one point. The solution is also applied to a gravitating spheroid of small excentricity, and to the rotation of a sphere or spherical shell.

Various theories as to rupture are explained, and expressions are obtained for the tendency to rupture in accordance with the two theories most extensively held. The application of these results is fully illustrated in the case of the problems last mentioned.

In the case of rotation it is found that the tendency to rupture in a solid sphere is greatest at the centre; while in a thin shell it is greatest along the inner surface in the equator, and is about four times as great as for a solid sphere of the same radius. In the earth, regarded as isotropic, it is found that, on either of the theories illustrated, the tendency to rupture due to diurnal rotation is greater than the tendency due to forces arising from the earth's spheroidal shape treated after the method of Professor Darwin*.

A solution is obtained for the vibrations of a sphere or spherical shell, giving the nature and magnitude of the vibrations due to the application of given periodic surface forces. The relation between the magnitude of the vibration and that of the constraining force, at least when purely normal, is shewn to depend only on the degree of the spherical surface harmonic giving the law of distribution of the force. By making the surface forces vanish equations are obtained giving the frequencies of the various forms of free vibration. Perfect agreement exists between the results so obtained and those otherwise investigated by Professor Lamb†, so far as the cases dealt with are the same.

The equations in cylindrical coordinates are treated in an analogous way, and general solutions obtained for the equilibrium or vibration of infinite solid or hollow cylinders under the action of given bodily or surface forces. It is also shewn how to extend these results to the case of coaxial cylindrical layers of different materials, and to the treatment of a material whose structure varies with the distance from an axis. Expressions for the tendency to rupture are worked out, and the conditions for their application illustrated in the case of the general solution.

Two solutions are also obtained applicable to various cases where the cylinder is of finite length. In the one expressions are obtained for the strains in ascending powers of z and r, measured respectively along and perpendicular to the axis, and this is applied to cylinders rotating under various conditions. It is found that in general the tendency to rupture in a rotating solid cylinder is much less than in a hollow cylinder of the same

* *Phil. Trans.* 1879.

† *Proc. London Mathematical Society*, Vol. XV.

external radius however small be the radius of the inner surface. The second solution embodies a general treatment of the equations, involving an extensive application of Bessel's and analogous functions. It is found usually to necessitate certain relations between the forces on the flat ends and on the curved surface, and its application to cases where the surface forces are arbitrarily assigned presents difficulties which have not been surmounted. It solves however certain problems of interest, the most important being the torsion of a cylinder when the forces causing torsion are any given function of the distance from the axis. The case when the force varies as r^3 is worked out as a special example.

(4) *A Table of the values of* e^x *for values of* x *between* 0 *and* 2 *increasing by* ·001. By F. W. NEWMAN, communicated by Prof. ADAMS, M.A.

Prof. Adams gave an account of the methods employed in constructing and verifying the Table which is being printed in full in the *Transactions* of the Society.

(5) *On the Application of Lagrange's Equations to the Motion of Perforated Solids in a Liquid when there is Circulation.* By A. B. BASSET, M.A.

1. When a number of perforated solids are moving in an infinite liquid, and there is circulation through the apertures of the solids, the kinetic energy of the solids and liquid (as will be proved later on) is equal to the sum of two homogeneous quadratic functions of the velocities and circulations respectively. Now when a liquid of density ρ occupies a multiply connected region, circulation κ can be generated by the application of a uniform impulsive pressure $\kappa\rho$, applied to any one of the barriers which must be drawn to render the region simply connected; and if once generated, it cannot be destroyed excepting by the same process as that by which it has been produced. It therefore appears that the quantity $\kappa\rho$ is a quantity in the nature of a generalised component of momentum.

Now the kinetic energy of a dynamical system is expressible in three forms, (i) a homogeneous quadratic function of the generalised velocities, which is the Lagrangian form; (ii) a similar function of the momenta, which is the Hamiltonian form; (iii) a mixed function of the momenta and velocities, which is called by Routh the modified form. The Lagrangian form is the only one which can be used in forming Lagrange's equations; and since

the kinetic energy in the present case is expressed in the mixed form (iii) it will be necessary to obtain a modified form of the Lagrangian function, which enables Lagrange's equations to be written down without a knowledge of the generalised velocity corresponding to the momentum $\kappa\rho$. The principal object of the paper is to determine this function in such a form, that it may be calculated whenever the velocity potential of the liquid is known; and the result is expressed in terms of the circulations, the velocities of the solids, and the surface integrals of the constituents of the velocity potential taken over the boundaries and barriers of the liquid. Finally it is shewn that the generalised velocity corresponding to $\kappa\rho$, where κ is the circulation round any closed circuit which cuts any particular aperture of one of the solids once only, is equal to the flux through that aperture *relative* to the particular solid*.

2. The following notation will be employed.

ϕ = velocity potential of the whole motion.

Ψ = do. due to motion of solids alone.

Ω = do. due to cyclic motion.

u_m, v_m, w_m; p_m, q_m, r_m, the linear and angular velocities respectively of any solid S_m along and about axes *fixed* in the solid.

ϕ_m', ϕ_m'', ϕ_m'''; χ_m', χ_m'', χ_m''', the velocity potentials of the liquid, when the solid S_m is moving with linear and angular velocities respectively along and about axes fixed in S_m, and all the other solids are at rest and there is no circulation.

σ_m, σ_m', σ_m''... the areas of the apertures of S_m.

κ_m, κ_m', κ_m''... the circulations through them.

ω_m, ω_m', ω_m''... the velocity potentials due to unit circulations through the apertures of S_m, when all the solids are at rest.

$\dot\psi_m$, $\dot\psi_m'$, $\dot\psi_m''$... the fluxes through the apertures of S_m *relative* to S_m.

Φ_m the velocity potential due to the motion of S_m and the circulations through its apertures, when all the other solids are at rest; so that

$$\Phi_m = u_m\phi'_m + v_m\phi''_m + w_m\phi'''_m + p_m\chi'_m + q_m\chi''_m + r_m\chi'''_m + \kappa_m\omega_m + \kappa'_m\omega'_m + \dots$$

* This theorem was discovered by Sir W. Thomson. *Proc. R. S. E.* Vol. VII. p. 668.

We know that the motion could be instantaneously produced from rest, by the application of suitable impulses to each of the solids and barriers. Let $\bar{X}_m$, $\bar{Y}_m$, Z_m; $\bar{L}_m$, $\bar{M}_m$, $\bar{N}_m$ be the force and couple components of the impulse along and about axes fixed in S_m, which must be applied to S_m.

Let ξ_m, η_m, ζ_m; λ_m, μ_m, ν_m; $\xi'_m, \eta'_m \dots$ be the components of the impulses which must be applied to each of the barriers of S_m; also let $\xi_m = \Sigma\xi_m$ &c.; $X_m = \bar{X}_m + \bar{\xi}_m$ &c., and let $\mathfrak{X}_m$, $\mathfrak{Y}_m$, $\mathfrak{Z}_m$; $\mathfrak{L}_m$, $\mathfrak{M}_m$, $\mathfrak{N}_m$ be the generalised components corresponding to $u_m, v_m \dots$ of the momentum of the cyclic motion, when all the solids are at rest.

Let M_m be the mass of S_m, $\mathfrak{T}$ the kinetic energy of the liquid, T that of the whole motion. It will be shewn that T is the sum of two homogeneous quadratic functions of the velocities and circulations respectively. Let these be denoted by $\mathfrak{I}$ and $\mathfrak{K}$ respectively, and let $(u_m u_m)$, $2(u_m v_m)$ denote the coefficients of u_m^2, $u_m v_m$ &c.

At the surface of S_m, $d\Phi_m/dn$ is equal to the normal velocity of S_m, and is zero at the surfaces of all the other solids; hence

$$\phi = \Sigma\Phi_m,$$

and therefore

$$2\mathfrak{T} = -\rho\iint\Phi\left(\frac{d\Phi_1}{dn}dS_1 + \frac{d\Phi_2}{dn}dS_2 + \dots\dots\right)$$
$$+\rho\iint\frac{d\phi}{dn}\left(\kappa_1 d\sigma_1 + \kappa_1' d\sigma_1' + \dots \kappa_2 d\sigma_2 + \dots\right).$$

Hence

$$\left.\begin{aligned}
(u_1u_1) &= -\rho\iint\phi_1'\frac{d\phi_1'}{dn}dS_1,\\
2(u_1u_2) &= -\rho\iint\phi_2'\frac{d\phi_1'}{dn}dS_1 - \rho\iint\phi_1'\frac{d\phi_2'}{dn}dS_2 = -2\rho\iint\phi_2'\frac{d\phi_1'}{dn}dS_1,\\
2(u_1\kappa_1) &= -\rho\iint\omega_1\frac{d\phi_1'}{dn}dS_1 + \rho\iint\frac{d\phi_1'}{dn}d\sigma_1 = 0,\\
2(u_1\kappa_2) &= -\rho\iint\omega_2\frac{d\phi_1'}{dn}dS_1 + \rho\iint\frac{d\phi_1'}{dn}d\sigma_2 = 0,\\
(\kappa_1\kappa_1) &= \rho\iint\frac{d\omega_1}{dn}d\sigma_1,\\
2(\kappa_1\kappa_1') &= \rho\iint\frac{d\omega_1'}{dn}d\sigma_1 + \rho\iint\frac{d\omega_1}{dn}d\sigma_1' = 2\rho\iint\frac{d\omega_1'}{dn}d\sigma_1,
\end{aligned}\right\} \quad \dots\dots\dots(1).$$

The above equations at once follow from Thomson's extension of Green's Theorem. For if in equations (30) and (31) of Lamb's Motion of Fluids, Art. 66, we put $\phi = \omega_1$ $\psi = \phi_1'$, then since ω_1 is a monocyclic function whose increment is unity for all circuits which cut the barrier σ_1 once, and zero for all other circuits, and ϕ_1' is a single valued function, we obtain

$$\iint \omega_1 \frac{d\phi_1'}{dn}\, dS - \iint \frac{d\phi_1'}{dn}\, d\sigma_1 = \iint \phi_1' \frac{d\omega_1}{dn}\, dS.$$

Now $d\phi_1'/dn$ is zero at the surfaces of all the solids except S_1, and $d\omega_1/dn$ is zero at the surfaces of all the solids (see Lamb, Art. 120), whence the third of equations (1) follows at once. The others can be proved in a similar manner; it therefore follows that

$$T = \mathfrak{T} + \mathfrak{K}$$

where $\mathfrak{T}$ is a homogeneous quadratic function of the velocities of the solids alone, and $\mathfrak{K}$ is a similar function of the circulations.

If p be the pressure and l_1, m_1, n_1 the direction cosines of the normal to S_1,

$$\begin{aligned} \bar{X}_1 &= M_1 u_1 + \iint p l_1 dS_1 \\ &= M_1 u_1 - \rho \iint \phi \frac{d\phi_1'}{dn}\, dS_1 . \end{aligned}$$

But

$$\begin{aligned} \frac{d\mathfrak{T}}{du_1} &= (u_1 u_1)\, u_1 + (u_1 v_1)\, v_1 + \\ &= -\rho \iint (u_1\phi_1' + v_1\phi_1'' + \quad) \frac{d\phi_1'}{dn}\, dS_1 \\ &= -\rho \iint \phi \frac{d\phi_1'}{dn}\, dS_1 + \rho \iint \Omega \frac{d\phi_1'}{dn}\, dS_1 \\ &= -\rho \iint \phi \frac{d\phi_1'}{dn}\, dS_1 + \rho \iint \frac{d\phi_1'}{dn} \Sigma\, (\kappa d\sigma), \end{aligned}$$

where the summation refers to corresponding products, and extends to all the barriers; hence

$$\bar{X}_1 = \frac{dT}{du_1} - \rho \iint \frac{d\phi_1'}{dn} \Sigma\, (\kappa d\sigma) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Therefore

$$X_1 = \bar{X}_1 + \bar{\xi}_1 = \frac{dT}{du_1} - \rho \iint \frac{d\phi_1'}{dn} \Sigma\, (\kappa d\sigma) + \bar{\xi}_1 \ldots\ldots\ldots (3),$$

where

$$\bar{\xi}_1 = \Sigma \xi_1 = \rho \iint \Sigma_1\, (\kappa l d\sigma),$$

and the summation Σ_1 extends to the barriers of S_1 only.

From (2) we see that the component impulse corresponding to u_1, which must be applied to S_1 in order to keep it at rest, when the cyclic motion is generated by the application of proper impulses to the barriers of all the solids is $-\rho\iint d\phi_1'/dn\,.\,\Sigma\,(\kappa d\sigma)$; and therefore by (3) the generalised component of momentum $\mathfrak{X}_1$ corresponding to u_1 of the cyclic motion when all the solids are reduced to rest, is

$$\mathfrak{X}_1 = \bar{\xi}_1 - \rho\iint\frac{d\phi_1'}{dn}\,\Sigma\,(\kappa d\sigma) = \rho\iint\Sigma_1\,(\kappa l d\sigma) - \rho\iint\frac{d\phi_1'}{dn}\,\Sigma\,(\kappa d\sigma)\ldots(4),$$

whence
$$X_1 = \frac{dT}{du_1} + \mathfrak{X}_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly
$$L_1 = \frac{dT}{dp_1} + \mathfrak{L}_1$$

where
$$\left.\begin{aligned}\mathfrak{L}_1 &= \bar{\lambda}_1 - \rho\iint\frac{d\chi_1'}{dn}\,\Sigma\,(\kappa d\sigma),\\ \bar{\lambda}_1 &= \Sigma\lambda_1 = \rho\iint\Sigma_1\,[\kappa\,(ny - mz)\,d\sigma]\end{aligned}\right\}\ldots\ldots\ldots(6).$$
and

3. We must now obtain an expression for the modified Lagrangian function.

Let the coordinates of a dynamical system be divided into two groups θ and χ, the latter of which does not enter into the expression for the energy of the system. Then it is known from the theory of ignoration of coordinates that,

$$\frac{dT}{d\dot{\chi}} = \text{const.} = \kappa \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and that if the velocities $\dot{\chi}$ be eliminated by means of the system of equations of (7) in the type, the resulting expression for T will be of the form

$$T = \mathfrak{T} - \mathfrak{P} + \mathfrak{K} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The quantities $\mathfrak{T}$, $\mathfrak{P}$, $\mathfrak{K}$ are defined as follows; let the original expression for T be

$$2T = (\theta\theta)\dot{\theta} + 2\,(\theta\theta_1)\,\dot{\theta}\dot{\theta}_1 + \ldots 2(\theta\chi)\,\dot{\theta}\dot{\chi} + \ldots (\chi\chi)\,\dot{\chi}^2 + 2(\chi\chi_1)\dot{\chi}\dot{\chi}_1 + \cdots$$

also let
$$\left.\begin{aligned}P &= (\theta\chi)\,\dot{\theta} + (\theta_1\chi)\,\dot{\theta}_1 + \\ P_1 &= (\theta\chi_1)\,\dot{\theta} + (\theta_1\chi_1)\,\dot{\theta}_1 +\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(9).$$

Then it is shewn in Thomson and Tait, Vol. I. Part I. pp. 320—323 (writing κ for C) that

$$\left.\begin{aligned}\dot{\chi} &= (\kappa\kappa)\,(\kappa - P) + (\kappa\kappa_1)(\kappa_1 - P_1) + \\ \dot{\chi}_1 &= (\kappa\kappa_1)(\kappa - P) + (\kappa_1\kappa_1)(\kappa_1 - P_1) +\end{aligned}\right\}\ldots\ldots\ldots(10),$$

and that
$$2\mathfrak{P} = (\kappa\kappa) P^2 + 2(\kappa\kappa_1) PP_1 + \dots$$
$$2\mathfrak{K} = (\kappa\kappa)\kappa^2 + 2(\kappa\kappa_1)\kappa\kappa_1 + \dots$$
where the coefficients are functions of the coordinates alone. Also $\mathfrak{T}$ is that portion of the original expression for T which does not contain the $\dot{\chi}$'s. Equations (10) may be written
$$\dot{\chi} = \frac{d\mathfrak{K}}{d\kappa} - \frac{d\mathfrak{P}}{dP}, \quad \dot{\chi}_1 = \frac{d\mathfrak{K}}{d\kappa_1} - \frac{d\mathfrak{P}}{dP_1} \text{ \&c.} \quad \dots\dots\dots\dots (11).$$

Let Θ be the generalised component of momentum corresponding to $\dot{\theta}$, and let $\bar{\Theta}$ be the value of Θ after the velocities $\dot{\theta}$ have been destroyed by means of proper impulses applied to the system. Then
$$\Theta = \frac{dT}{d\dot{\theta}} = \frac{d\mathfrak{T}}{d\dot{\theta}} + (\theta\chi)\dot{\chi} + (\theta\chi_1)\dot{\chi}_1 +$$
$$= \frac{d\mathfrak{T}}{d\dot{\theta}} + (\theta\chi)\left(\frac{d\mathfrak{K}}{d\kappa} - \frac{d\mathfrak{P}}{dP}\right) +$$
whence
$$\bar{\Theta} = (\theta\chi)\frac{d\mathfrak{K}}{d\kappa} + (\theta\chi_1)\frac{d\mathfrak{P}}{d\kappa_1} + \dots\dots\dots\dots (12).$$

Let
$$\mathfrak{X} = \mathfrak{T} - \mathfrak{P},$$
then
$$\frac{dT}{d\dot{\theta}} = \frac{d\mathfrak{T}}{d\dot{\theta}} + \bar{\Theta} - \Sigma(\theta\chi)\frac{d\mathfrak{P}}{dP}$$
$$= \frac{d\mathfrak{T}}{d\dot{\theta}} + \bar{\Theta} - \Sigma\frac{dP}{d\dot{\theta}}\frac{d\mathfrak{P}}{dP}$$
$$= \frac{d\mathfrak{X}}{d\dot{\theta}} + \bar{\Theta} \quad \dots\dots\dots\dots (13),$$
whence
$$\frac{d}{dt}\frac{dT}{d\dot{\theta}} = \frac{d}{dt}\frac{d\mathfrak{X}}{d\dot{\theta}} + \frac{d\bar{\Theta}}{dt} \quad \dots\dots\dots\dots (14).$$

The momentum $\bar{\Theta}$ is evidently a function of the momenta κ and the coordinates only.

Again
$$\frac{dT}{d\theta} = \frac{d\mathfrak{X}}{d\theta} + \frac{d\mathfrak{K}}{d\theta}.$$

Now since θ enters into $\mathfrak{K}$ through κ, we have
$$\frac{d\mathfrak{K}}{d\theta} = \frac{d\mathfrak{K}}{d\kappa}\frac{d\kappa}{d\theta} + \frac{d\mathfrak{K}}{d\kappa_1}\frac{d\kappa_1}{d\theta} + \dots \frac{\mathfrak{d}\mathfrak{K}}{d\theta},$$
where the symbol $\mathfrak{d}/d\theta$ operates on the coefficients and not on the momenta κ. Differentiating (10) with respect to θ, we obtain

$$0 = (\kappa\kappa)\left(\frac{d\kappa}{d\theta} - \frac{dP}{d\theta}\right) + (\kappa\kappa_1)\left(\frac{d\kappa_1}{d\theta} + \frac{dP_1}{d\theta}\right) +$$
$$+ (\kappa - P)\frac{d}{d\theta}(\kappa\kappa) + (\kappa_1 - P_1)\frac{d}{d\theta}(\kappa\kappa_1) + \ldots\ldots (15).$$

Multiplying the system of equations of which (15) is the type by κ, κ_1... respectively and adding we obtain

$$\frac{d\mathfrak{K}}{d\kappa}\frac{d\kappa}{d\theta} + \frac{d\mathfrak{K}}{d\kappa_1}\frac{d\kappa_1}{d\theta} + \ldots\ldots$$
$$- \frac{d\mathfrak{K}}{d\kappa}\frac{dP}{d\theta} - \frac{d\mathfrak{K}}{d\kappa_1}\frac{dP_1}{d\theta} - \ldots\ldots$$
$$+ 2\frac{\mathfrak{d}\mathfrak{K}}{d\theta} - P\frac{\mathfrak{d}}{d\theta}\frac{d\mathfrak{K}}{d\kappa} - P_1\frac{\mathfrak{d}}{d\theta}\cdot\frac{d\mathfrak{K}}{d\kappa_1} - \ldots = 0 \quad \ldots\ldots \quad (16).$$

Multiplying the equations of which (12) is the type by $\dot{\theta}$, $\dot{\theta}_1$... respectively and adding, we obtain

$$\Sigma(\overline{\Theta}\dot{\theta}) = P\frac{d\mathfrak{K}}{d\kappa} + P_1\frac{d\mathfrak{K}}{d\kappa_1} +$$

whence $$\frac{\mathfrak{d}}{d\theta}\Sigma(\overline{\Theta}\dot{\theta}) = \Sigma\left(\frac{dP}{d\theta}\frac{d\mathfrak{K}}{d\kappa}\right) + \Sigma\left(P\frac{\mathfrak{d}}{d\theta}\cdot\frac{d\mathfrak{K}}{d\kappa}\right),$$

therefore (16) becomes $$\frac{d\mathfrak{K}}{d\theta} + \frac{\mathfrak{d}\mathfrak{K}}{d\theta} - \frac{\mathfrak{d}}{d\theta}\Sigma(\overline{\Theta}\dot{\theta}) = 0,$$

whence $$\frac{dT}{d\theta} = \frac{d\mathfrak{T}}{d\theta} - \frac{\mathfrak{d}\mathfrak{K}}{d\theta} + \frac{\mathfrak{d}}{d\theta}\Sigma\,(\overline{\Theta}\dot{\theta}).$$

We may now drop the symbol $\mathfrak{d}/d\theta$ on the understanding that the momenta κ are to be treated as constants, and Lagrange's equations become

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{\theta}} + \frac{d\overline{\Theta}}{dt} - \frac{d\mathfrak{T}}{d\theta} + \frac{d\mathfrak{K}}{d\theta} - \frac{d}{d\theta}\Sigma\,(\overline{\Theta}\dot{\theta}) + \frac{dV}{d\theta} = 0.$$

Since $\overline{\Theta}$ and $\mathfrak{K}$ do not contain $\dot{\theta}$, the modified function is

$$L' = \mathfrak{T} + \Sigma\,(\Theta\dot{\theta}) - \mathfrak{K} + V \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

If the velocities $\dot{\theta}$... be expressed in terms of new velocities u..., and $\mathfrak{X}$ be the new momentum corresponding to u after the u's have been destroyed, it can easily be shewn that,

$$\Sigma\,(\overline{\Theta}\dot{\theta}) = \Sigma\,(\mathfrak{X}u).$$

For let $$\dot{\theta} = Au + A_1u_1 + A_2u_2 +$$

then $$d\dot{\theta}/du = A, \quad d\dot{\theta}/du_1 = A_1, \text{ \&c.}$$

also by (13), $$\Sigma\left(\overline{\Theta}\frac{d\dot{\theta}}{du}\right) = \frac{dT^*}{du} - \frac{d\mathfrak{T}}{du} = \mathfrak{X}\cdot$$

* In this term T is supposed to be expressed in terms of the velocities u. . and $\dot{\chi}$....

therefore

$$\Sigma(\mathfrak{X}u) = \bar{\Theta}\left(u\frac{d\theta}{du} + u_1\frac{d\dot{\theta}}{du_1} + \right) + \bar{\Theta}_1\left(u\frac{d\dot{\theta}_1}{du} + \right) + \&c.$$
$$= \Sigma(\bar{\Theta}\dot{\theta})$$

whence (17) may be written

$$L' = \mathfrak{T} + \Sigma(\mathfrak{X}u) - \mathfrak{K} + V \quad \text{..............} (18).$$

4. We have therefore obtained a form of Lagrange's equations, which can be employed when the kinetic energy is expressed in terms of the velocities corresponding to the coordinates by which the position of the system is determined, and the constant momenta corresponding to the time fluxes of the ignored coordinates. Now in the hydrodynamical problem we are considering, the product of the circulations and density of the liquid corresponds to the constant momenta κ, κ_1.... Hence in order to determine the motion of a number of perforated solids in an infinite liquid, we must first calculate by means of (1) the quantities $\mathfrak{T}$ and $\mathfrak{K}$; the former of which is the kinetic energy due to the motion of the solids alone, and is therefore a homogeneous quadratic function of their velocities, and must be expressed in terms of the generalised coordinates and velocities of each solid; and the latter of which is a similar function of the circulations. The quantity $\mathfrak{X}$ in (18) is evidently the generalised component corresponding to u, of the momentum of the cyclic motion which remains after all the solids have been reduced to rest, and its value is given by (4) or (6), according as it is in the nature of a force or a couple.

5. We can now ascertain the physical meaning of the generalised velocity $\dot{\chi}$ which corresponds to the momentum $\kappa\rho$.

Let $\dot{\psi}_1$ be the flux through the aperture σ_1 of S_1 relative to S_1. Then if l_1, m_1, n_1 be the direction cosines of the normal to σ_1

$$\dot{\psi}_1 = \iint\left\{\frac{d\phi}{dn} - l_1(u_1 + q_1 z - r_1 y) - m_1(v_1 + r_1 x - p_1 z) \right.$$
$$\left. - n_1(w_1 + p_1 y - q_1 x)\right\} d\sigma_1$$
$$= \iint\frac{d\Psi}{dn}d\sigma_1 + \iint\frac{d\Omega}{dn}d\sigma_1 - (u_1\xi_1 + v_1\eta_1 + w_1\zeta_1 + p_1\lambda_1 + q_1\mu_1 + r_1\nu_1)/\kappa_1\rho.$$

But
$$\rho\iint\frac{d\Omega}{dn}d\sigma_1 = (\kappa_1\kappa_1)\,\kappa_1 + (\kappa_1\kappa_1')\,\kappa_1' +$$
$$= \frac{d\mathfrak{K}}{d\kappa_1}.$$

If therefore we put

$$\alpha_1 = \xi_1/\kappa_1\rho = \iint l_1 d\sigma_1, \quad \beta_1 = \eta_1/\kappa_1\rho, \quad \gamma_1 = \zeta_1/\kappa_1\rho,$$

$$a_1 = \lambda_1/\kappa_1\rho = \iint (n_1 y - m_1 z)\, d\sigma_1, \quad b_1 = \mu_1/\kappa_1\rho, \quad c_1 = \nu_1/\kappa_1\rho,$$

we obtain

$$\dot{\psi}_1 = \iint \frac{d\Psi}{dn}\, d\sigma_1 - \alpha_1 u_1 - \beta_1 v_1 - \gamma_1 w_1 - a_1 p_1 - b_1 q_1 - c_1 r_1 + \frac{1}{\rho}\frac{d\mathfrak{K}}{d\kappa_1} \quad\text{.........(19).}$$

Now if T be expressed as a quadratic function of all the momenta

$$\frac{1}{\rho}\frac{dT}{d\kappa} = \dot{\chi}.$$

But $$2T = \Sigma u \frac{d\mathfrak{T}}{du} + 2\mathfrak{K} = \Sigma u (X - \mathfrak{X}) + 2\mathfrak{K} \quad\text{......... (20),}$$

by (5). Hence in order to obtain $\dot{\chi}_1$ we must differentiate (20) with respect to κ_1, on the hypothesis that the momenta X are constant, and that u is a function of κ_1; whence by (4) and (6),

$$2\frac{dT}{d\kappa_1} = \Sigma \frac{d\mathfrak{T}}{du}\frac{du}{d\kappa_1} - (\alpha_1 u_1 + \beta_1 v_1 + \gamma_1 w_1 + a_1 p_1 + b_1 q_1 + c_1 r_1)\rho + \rho \iint \frac{d\Psi}{dn}\, d\sigma_1 + 2\frac{d\mathfrak{K}}{d\kappa_1} \quad\text{.................. (21).}$$

From (5) we obtain

$$0 = \Sigma\, (u_1 v) \frac{dv}{d\kappa_1} + \alpha_1\rho - \rho \iint \frac{d\phi_1'}{dn}\, d\sigma_1,$$

....

$$0 = \Sigma\, (u_2 v) \frac{dv}{d\kappa_1} - \rho \iint \frac{d\phi_2'}{dn}\, d\sigma_2,$$

....

where the summation extends to all the unsuffixed letters including $v = u_1$. Multiplying these equations by u_1, v_1... respectively and adding we obtain

$$\Sigma \frac{d\mathfrak{T}}{du}\frac{du}{d\kappa_1} + (\alpha_1 u_1 + \beta_1 v_1 + \gamma_1 w_1 + a_1 p_1 + b_1 q_1 + c_1 r_1)\,\rho - \rho \iint \frac{d\Psi}{dn}\, d\sigma_1 = 0,$$

whence by (19) and (21) $$\frac{dT}{d\kappa_1} = \rho\dot{\psi}_1$$

whence $$\dot{\chi}_1 = \dot{\psi}_1.$$

Hence the flux through the aperture σ_1 relative to the solid S_1 is the generalised velocity corresponding to the momentum $\kappa_1\rho$.

6. As an example of the above formulæ, consider the motion of an anchor ring discussed ante p. 47.

Here $$L' = \tfrac{1}{2}(Pu^2 + Rw^2 + A\dot{\theta}^2) + \zeta_0 w - \tfrac{1}{2}K\kappa^2,$$
$$u = \dot{x}\cos\theta - \dot{z}\sin\theta,$$
$$w = \dot{x}\sin\theta + \dot{z}\cos\theta.$$

Since L' does not contain x or z, we have

$$\frac{dL'}{d\dot{x}} = \text{const.}; \quad \frac{dL'}{d\dot{z}} = \text{const.};$$

whence $$Pu\cos\theta + Rw\sin\theta + \zeta_0\sin\theta = E$$
$$-Pu\sin\theta + Rw\cos\theta + \zeta_0\cos\theta = F.$$

Since $u = 0$, $w = 0$ when $\theta = 0$; we obtain $E = 0$, $F = \zeta_0$,

therefore $$\left.\begin{aligned} Pu &= -\zeta_0\sin\theta \\ Rw &= -\zeta_0(1-\cos\theta) \end{aligned}\right\} \quad\text{..................(22).}$$

Also $$\frac{dL'}{d\theta} = (R-P)\,uw + \zeta_0 u,$$

whence $$A\ddot{\theta} + \zeta_0^2\left(\frac{1}{P} - \frac{1}{R}\right)\sin\theta\cos\theta + \frac{\zeta_0^2}{R}\sin\theta = 0.$$

This is the equation which is obtained by eliminating u and w from the equation of energy by means of (22), and differentiating the result with respect to the time.

[7. The expression for L' in the form obtained in § 3. (18) cannot always be employed to determine the two dimensional motion of a number of cylinders round which there is circulation, owing to the fact that it sometimes happens that in this case some of the terms are infinite. The adaptation of the formula to this problem, and its application to determine the motion of a cylinder moving in a liquid bounded by a fixed plane, will be dealt with in a paper shortly to be communicated to the Society. Nov. 18th.]

Dec. 11th. I take the opportunity of correcting two errors in my former paper "On the Motion of a Ring."

On p. 55 it is stated that the two values of p^2 are negative. This is necessarily the case if $Z\gamma$ is positive; but it is possible for $Z\gamma$ to be negative, either by reason of the ring being prolate, or by reason of its velocity being in the opposite direction to that of the cyclic motion through its aperture. In either of these cases the coefficient of p^2 will be negative, and the motion will be unstable unless Ω be sufficiently great.

On p. 60. The statement that both positions of steady motion are stable, is incorrect.

The condition of stability is, that the right-hand side of the last equation should be positive. This requires that

$$\left\{\frac{2\zeta_0}{R} - Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 + 3\cos\alpha)\right\}\left\{\frac{2\zeta_0}{R} + Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 - 3\cos\alpha)\right\} > 0.$$

If there is no circulation $\zeta_0 = 0$, and the condition is that $\cos\alpha$ should be greater than $\tfrac{1}{3}$ or less than $-\tfrac{1}{3}$.

November 14, 1887.

(1) *On the Fungus causing the onion disease Peronospora Schleideniana.* By A. E. Shipley, B.A.

The onion disease of the Bermudas is caused by a fungus *Peronospora Schleideniana*, which lives parasitically upon the leaf of the onion plant.

The atmospheric conditions which favour the progress of the disease are heavy dews or rains followed by warm, moist, calm weather, and the absence of direct sunshine and cold winds. In favourable weather the progress of the disease is very rapid.

The fungus lives in the tissues of the leaf, choking up the air passages and absorbing the nutritive fluid formed in the cells. Its stem protrudes through the stomata of the leaf into the air. Its branches bear spores at their tips.

The reproduction of the fungus is effected by means of these spores which float about through the air, and also by means of certain special cells formed by the fungus and known as resting-spores. These pass the winter in the earth, and are capable of retaining the power of germination for two or three years. It is by their means that the disease is carried on from one season to another.

One method of combating the disease is to make the onion plants as strong as possible, so as to withstand the attacks of the parasite. Hence the site should be carefully selected, the soil well prepared, good manures used, and the land kept clean and free from weeds.

To prevent the spreading of the disease all affected plants must be collected and burnt. Whilst doing this care must be taken that the collector does not himself spread the disease by carrying the refuse loosely. Rotation of crops, or, when this is impossible, deep trenching, would lessen the chance of the disease appearing.

Diseased plants may be treated with a mixture of powdered sulphur and freshly burnt quicklime sprinkled by hand or by bellows; or they may be washed or sprayed with a weak solution of iron sulphate (green vitriol). In both cases the fungus is destroyed without injury to the onion plant. Further, both these chemical remedies have the additional advantage of being excellent manures.

Another fungus *Macrosporium parasiticum* sometimes attacks the onions after the *Peronospora* has taken a good hold of the plant and weakened it. As this only occurs as a sequel to the

Peronospora, the extermination of the latter would involve the disappearance of the former. The *Macrosporium* does not attack the healthy plant.

Only two kinds of insects, the onion thrips and the onion fly, were met with, and the latter on only one occasion. The thrips were not numerous and appeared to do little harm. They can easily be removed by application of a solution of iron sulphate, such as is recommended above.

The onion fly may be dealt with by covering the bulb of the onion with a thin layer of earth. This prevents the fly approaching the bulb to lay its eggs.

(2) *On Alternation of Generations in Green Plants.* By J. Reynolds Vaizey, B.A.

The objects of this paper are to discuss the origin of Alternation of Generations in green plants, and to consider what effect the view of the origin of alternation of generations has on comparisons between the vegetative bodies of the oophyte and sporophyte of the same or different forms.

Comparisons of the life-histories of *Coleochæte*, *Oedogonium*, *Sphaeroplea*, *Ulothrix*, *Hydrodictyon*, *Pandorina* and *Chara*, with that of the lowest mosses shew that in all these forms there is virtually alternation of generations. In the lowest forms the sporophyte is shewn to consist of a simple mass of cells produced by division of the oospore, each cell becoming sooner or later a spore which gives rise to the oophyte. Upon these comparisons it is suggested that alternation of generations arose from polyembryony, not, as according to Pringsheim's theory, by a process of differentiation from a number of individuals which were both sexual and asexual.

If this hypothesis is true, it is then pointed out that the sporophyte is a new body originating among the higher Algae and Liverworts, not genetically connected with the sexual body. Consequently the tissues of the sporophore cannot be homologous although they may be analogous with those of the oophyte.

(3) *On a new species of spider, with some observations on the habits of certain Araneina.* By C. Warburton.

The new spider was of the genus *Linyphia*, and nearly allied to *Linyphia rufa* Westr. from which, however, it was readily distinguishable by the structure of the male palpi, as well as by other peculiarities.

Its habitat was Southport, Lancashire, where it was taken in the "slacks," or swampy districts among the sandhills.

The paper also dealt with the fertilization of the female in Araneina, and recorded an observation on this head with regard to the species *Neriene rubens*.

Some points in the structure of the nets of orb-weavers were commented on, and an instance adduced illustrative of the possession of a certain amount of intelligence by spiders.

Some typical British spiders were exhibited, together with a few specimens from Bermudas collected by Mr A. E. Shipley, Fellow of Christ's College.

(4) *On expressions for the Theta Functions as Definite Integrals.* (Second paper.) By J. W. L. GLAISHER, Sc. D., F.R.S.

§ 1. In Vol. III. (1877) pp. 61—66 an account was given of four different methods of obtaining expressions for the Theta functions as definite integrals. I have recently worked out the expressions for the four Theta functions by the third and fourth of the methods described in that paper.

Putting $\mu = \frac{\pi K'}{K}$, $\rho = \frac{2K}{\pi}$, and $u = \rho x = \frac{2Kx}{\pi}$, and denoting by $\Theta(u)$, $\Theta_1(u)$, $\Theta_2(u)$, $\Theta_3(u)$ the four Theta functions $\Theta(u)$, $H(u)$, $H(u+K)$, $\Theta(u+K)$, we have

$$\Theta(u) = \frac{\sqrt{\pi}}{\sqrt{\mu}} \Sigma_{-\infty}^{\infty} \quad e^{-\frac{(x+l\pi)^2}{\mu}},$$

$$\Theta_1(u) = -\frac{\sqrt{\pi}}{\sqrt{\mu}} \Sigma_{-\infty}^{\infty} (-)^n e^{-\frac{(x+l\pi)^2}{\mu}},$$

$$\Theta_2(u) = \frac{\sqrt{\pi}}{\sqrt{\mu}} \Sigma_{-\infty}^{\infty} (-)^n e^{-\frac{(x+n\pi)^2}{\mu}},$$

$$\Theta_3(u) = \frac{\sqrt{\pi}}{\sqrt{\mu}} \Sigma_{-\infty}^{\infty} \quad e^{-\frac{(x+n\pi)^2}{\mu}},$$

where l denotes $n + \frac{1}{2}$.

§ 2. By applying the third method, described in § 6 of the previous paper, to the summation of these series, I have found that

$$\Theta(u) = \frac{2}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh 2t - \sin 2(t+x)}{\cosh 2t + \cos 2(t+x)} + \frac{\sinh 2t - \sin 2(t-x)}{\cosh 2t + \cos 2(t-x)} \right\} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{2}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh 2t + \sin 2(t+x)}{\cosh 2t + \cos 2(t+x)} + \frac{\sinh 2t + \sin 2(t-x)}{\cosh 2t + \cos 2(t-x)} \right\} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

$$\Theta_1(u) = \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh t \sin(t+x) - \cosh t \cos(t+x)}{\cosh 2t + \cos 2(t+x)} - \frac{\sinh t \sin(t-x) - \cosh t \cos(t-x)}{\cosh 2t + \cos 2(t-x)} \right\} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh t \sin(t+x) + \cosh t \cos(t+x)}{\cosh 2t + \cos 2(t+x)} - \frac{\sinh t \sin(t-x) + \cosh t \cos(t-x)}{\cosh 2t + \cos 2(t-x)} \right\} \cos\left(\frac{2t^2}{\mu}\right) dt$$

$$\Theta_2(u) = \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh t \cos(t+x) + \cosh t \sin(t+x)}{\cosh 2t - \cos 2(t+x)} + \frac{\sinh t \cos(t-x) + \cosh t \sin(t-x)}{\cosh 2t - \cos 2(t-x)} \right\} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh t \cos(t+x) - \cosh t \sin(t+x)}{\cosh 2t - \cos 2(t+x)} + \frac{\sinh t \cos(t-x) - \cosh t \sin(t-x)}{\cosh 2t - \cos 2(t-x)} \right\} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

$$\Theta_3(u) = \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \left\{ \frac{\sinh 2t + \sin 2(t+x)}{\cosh 2t - \cos 2(t+x)} + \frac{\sinh 2t + \sin 2(t-x)}{\cosh 2t - \cos 2(t-x)} \right\} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{4}{\sqrt{\pi\mu}} \int^\infty \left\{ \frac{\sinh 2t - \sin 2(t+x)}{\cosh 2t - \cos 2(t+x)} + \frac{\sinh 2t - \sin 2(t-x)}{\cosh 2t - \cos 2(t-x)} \right\} \cos\left(\frac{2t^2}{\mu}\right) dt.$$

These expressions put in evidence the periodicity of th functions with respect to 2π, the argument x entering into the merely in the form of a quantity added to, or subtracted from, th variable t in the circular functions.

§ 3. Similarly, by applying the fourth method described in § 7 of the previous paper, to the q-series for $\Theta(u)$ and $\Theta_1(u)$, it is found that

$$\Theta(u) = 2\int_0^\infty \frac{A+B+C+D}{\cosh \pi t - \cos \pi t} \sin(\tfrac{1}{2}\mu t^2)\, dt$$

$$= 1 + 2\int_0^\infty \frac{A+B-C-D}{\cosh \pi t - \cos \pi t} \cos(\tfrac{1}{2}\mu t^2)\, dt,$$

$$\Theta_1(u) = 2\int_0^\infty \frac{A-B+C-D}{\cosh \pi t + \cos \pi t} \sin(\tfrac{1}{2}\mu t^2)\, dt$$

$$= 2\int_0^\infty \frac{A-B-C+D}{\cosh \pi t + \cos \pi t} \cos(\tfrac{1}{2}\mu t^2)\, dt*,$$

where
$$A = \sinh(\tfrac{1}{2}\pi + x)\, t \cos(\tfrac{1}{2}\pi - x)\, t,$$
$$B = \sinh(\tfrac{1}{2}\pi - x)\, t \cos(\tfrac{1}{2}\pi + x)\, t,$$
$$C = \cosh(\tfrac{1}{2}\pi + x)\, t \sin(\tfrac{1}{2}\pi - x)\, t,$$
$$D = \cosh(\tfrac{1}{2}\pi - x)\, t \sin(\tfrac{1}{2}\pi + x)\, t.$$

In these formulæ x must not exceed the limits $\pm\frac{1}{2}\pi$.

The functions $\Theta_2(u)$ and $\Theta_3(u)$ are represented by exactly the same expressions as $\Theta_1(u)$ and $\Theta(u)$ respectively, if we now suppose A, B, C, D to have the values

$$A = \sinh(\pi - x)\, t \cos xt,$$
$$B = \sinh xt \cos(\pi - x)\, t,$$
$$C = \cosh(\pi - x)\, t \sin xt,$$
$$D = \cosh xt \sin(\pi - x)\, t.$$

These formulæ for $\Theta_2(u)$ and $\Theta_3(u)$ hold good when x does not exceed the limits 0 and π.

§ 4. Since $\Theta_3(0) = \sqrt{\rho}$, $\Theta(0) = \sqrt{k'\rho}$, $\Theta_2(0) = \sqrt{k\rho}$, and $\Theta_1(\rho x) = \sqrt{kk'\rho^3}x$ when x is very small, we find by putting x

* In these formulæ for $\Theta(u)$ and $\Theta_1(u)$ we may express the four numerators $A+B+C+D$, $A+B-C-D$, &c. also in the forms

$$2(P+Q),\ 2(P-Q),\ 2(P'-Q'),\ 2(P'+Q')$$

respectively, where

$$P = \sinh\tfrac{1}{2}\pi t \cos\tfrac{1}{2}\pi t \cosh xt \cos xt + \cosh\tfrac{1}{2}\pi t \sin\tfrac{1}{2}\pi t \sinh xt \sin xt,$$
$$Q = \cosh\tfrac{1}{2}\pi t \sin\tfrac{1}{2}\pi t \cosh xt \cos xt - \sinh\tfrac{1}{2}\pi t \cos\tfrac{1}{2}\pi t \sinh xt \sin xt,$$
$$P' = \sinh\tfrac{1}{2}\pi t \sin\tfrac{1}{2}\pi t \cosh xt \sin xt + \cosh\tfrac{1}{2}\pi t \cos\tfrac{1}{2}\pi t \sinh xt \cos xt,$$
$$Q' = \cosh\tfrac{1}{2}\pi t \cos\tfrac{1}{2}\pi t \cosh xt \sin xt - \sinh\tfrac{1}{2}\pi t \sin\tfrac{1}{2}\pi t \sinh xt \cos xt.$$

equal to zero in the formulæ for $\Theta_3(x)$, $\Theta(x)$, $\Theta_2(x)$ and making x very small in the formula for $\Theta_1(x)$,

$$\sqrt{\rho} = \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh 2t + \sin 2t}{\cosh 2t - \cos 2t} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{\sqrt{\pi}}{\sqrt{\mu}} + \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh 2t - \sin 2t}{\cosh 2t - \cos 2t} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

$$\sqrt{k'\rho} = \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh 2t - \sin 2t}{\cosh 2t + \cos 2t} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{4}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh 2t + \sin 2t}{\cosh 2t + \cos 2t} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

$$\sqrt{k\rho} = \frac{8}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh t \cos t + \cosh t \sin t}{\cosh 2t - \cos 2t} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{\sqrt{\pi}}{\sqrt{\mu}} + \frac{8}{\sqrt{\pi\mu}} \int_0^\infty \frac{\sinh t \cos t - \cosh t \sin t}{\cosh 2t - \cos 2t} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

$$\sqrt{kk'\rho^3} = \frac{8}{\sqrt{\pi\mu}} \int_0^\infty \frac{(Cc+Ss)\sinh 2t - (Cc-Ss)\sin 2t}{(\cosh 2t + \cos 2t)^2} \sin\left(\frac{2t^2}{\mu}\right) dt$$

$$= \frac{8}{\sqrt{\pi\mu}} \int_0^\infty \frac{(Cc-Ss)\sinh 2t + (Cc+Ss)\sin 2t}{(\cosh 2t + \cos 2t)^2} \cos\left(\frac{2t^2}{\mu}\right) dt,$$

where, for brevity, S, C, s, c are used to denote

$$\sinh t, \quad \cosh t, \quad \sin t, \quad \cos t.$$

The occurrence of the term $\dfrac{\sqrt{\pi}}{\sqrt{\mu}}$ in the second of the formulæ for $\sqrt{\rho}$ and $\sqrt{k\rho}$ is due to the fact that the integral

$$\int_0^\infty \frac{2x^2}{t^2 + x^4} \cos\left(\frac{t}{\mu}\right) dt$$

is no longer equal to $\pi e^{-\frac{x^2}{\mu}}$ when x is zero, the value of the integral being then zero. If therefore we put $x = 0$ in the general formulæ we must, corresponding to the term $\dfrac{\sqrt{\pi}}{\sqrt{\mu}} e^{-\frac{x^2}{\mu}}$ in the series, introduce separately the term $\dfrac{\sqrt{\pi}}{\sqrt{\mu}}$, as this term is not included in the value of the integral expression.

§ 5. In the formulæ of the last section we may substitute in the right-hand members $\frac{\pi^2}{\mu}$ for μ, and multiply by $\frac{\sqrt{\pi}}{\sqrt{\mu}}$, if at the same time we interchange on the left-hand side $\sqrt{k\rho}$ and $\sqrt{k'\rho}$. This transformation corresponds to the change of modulus from k to k', for by this change μ is converted into $\frac{\pi^2}{\mu}$, and the multiplier $\frac{\sqrt{\pi}}{\sqrt{\mu}}$ is equal to $\frac{\sqrt{\rho}}{\sqrt{\rho'}}$. We thus find

$$\sqrt{\rho} = \frac{4}{\pi}\int_0^\infty \frac{\sinh 2t + \sin 2t}{\cosh 2t - \cos 2t} \sin\left(\frac{2\mu t^2}{\pi^2}\right) dt$$

$$= 1 + \frac{4}{\pi}\int_0^\infty \frac{\sinh 2t - \sin 2t}{\cosh 2t - \cos 2t} \cos\left(\frac{2\mu t^2}{\pi^2}\right) dt,$$

$$\sqrt{k'\rho} = \frac{8}{\pi}\int_0^\infty \frac{\sinh t \cos t + \cosh t \sin t}{\cosh 2t - \cos 2t} \sin\left(\frac{2\mu t^2}{\pi^2}\right) dt$$

$$= 1 + \frac{8}{\pi}\int_0^\infty \frac{\sinh t \cos t - \cosh t \sin t}{\cosh 2t - \cos 2t} \cos\left(\frac{2\mu t^2}{\pi^2}\right) dt,$$

$$\sqrt{k\rho} = \frac{4}{\pi}\int_0^\infty \frac{\sinh 2t - \sin 2t}{\cosh 2t + \cos 2t} \sin\left(\frac{2\mu t^2}{\pi^2}\right) dt$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sinh 2t + \sin 2t}{\cosh 2t + \cos 2t} \cos\left(\frac{2\mu t^2}{\pi^2}\right) dt,$$

$$\sqrt{kk'\rho^3} = \frac{8}{\pi}\int_0^\infty \frac{(Cc + Ss)\sinh 2t - (Cc - Ss)\sin 2t}{(\cosh 2t + \cos 2t)^2} \sin\left(\frac{2\mu t^2}{\pi^2}\right) dt$$

$$= \frac{8}{\pi}\int_0^\infty \frac{(Cc - Ss)\sinh 2t + (Cc + Ss)\sin 2t}{(\cosh 2t + \cos 2t)^2} \cos\left(\frac{2\mu t^2}{\pi^2}\right) dt.*$$

§ 6. The factor $\frac{1}{\sqrt{(\pi\mu)}}$ by which the integrals in § 4 were multiplied may be removed by transforming them by the substitution $t = \sqrt{\pi\mu}\,\tau$, and the factor $\frac{1}{\pi}$ may be removed from the integrals in the last section by putting $t = \pi\tau$.

* We may obtain the formulæ in this section directly by giving to x its limiting values in § 3 and transforming the integrals by the substitution $\pi t = 2\tau$.

Thus, for example,

$$\sqrt{\rho} = 4\int_0^\infty \frac{\sinh 2\sqrt{\pi\mu t} + \sin 2\sqrt{\pi\mu t}}{\cosh 2\sqrt{\pi\mu t} - \cos 2\sqrt{\pi\mu t}} \sin(2\pi t^2)\, dt$$

$$= 4\int_0^\infty \frac{\sinh 2\pi t + \sin 2\pi t}{\cosh 2\pi t - \cos 2\pi t} \sin(2\mu t^2)\, dt,$$

and also

$$= \frac{\sqrt{\pi}}{\sqrt{\mu}} + 4\int_0^\infty \frac{\sinh 2\sqrt{\pi\mu}\,t - \sin 2\sqrt{\pi\mu}\,t}{\cosh 2\sqrt{\pi\mu}\,t - \cos 2\sqrt{\pi\mu}\,t} \cos(2\pi t^2)\, dt$$

$$= 1 + 4\int_0^\infty \frac{\sinh 2\pi t - \sin 2\pi t}{\cosh 2\pi t - \cos 2\pi t} \cos(2\mu t^2)\, dt.$$

§ 7. It may be remarked that these formulæ for $\sqrt{\rho}$ are also of interest on account of their affording summations of the geometrical series of the second order as definite integrals, viz., we have

$$\Sigma_1^\infty x^{n^2} = -\tfrac{1}{2} + 2\int_0^\infty \frac{\sinh 2\sqrt{\pi a}\,t + \sin 2\sqrt{\pi a}\,t}{\cosh 2\sqrt{\pi a}\,t - \cos 2\sqrt{\pi a}\,t} \sin(2\pi t^2)\, dt$$

$$= -\tfrac{1}{2} + 2\int_0^\infty \frac{\sinh 2\pi t + \sin 2\pi t}{\cosh 2\pi t - \cos 2\pi t} \sin(2at^2)\, dt$$

$$= \tfrac{1}{2}\,\frac{\sqrt{\pi} - \sqrt{a}}{\sqrt{a}} + 2\int_0^\infty \frac{\sinh 2\sqrt{\pi a}\,t - \sin 2\sqrt{\pi a}\,t}{\cosh 2\sqrt{\pi a}\,t - \cos 2\sqrt{\pi a}\,t} \cos(2\pi t^2)\, dt$$

$$= 2\int_0^\infty \frac{\sinh 2\pi t - \sin 2\pi t}{\cosh 2\pi t - \cos 2\pi t} \cos(2at^2)\, dt.$$

§ 8. In the previous paper in Vol. III. of the *Proceedings* I gave the value of $\Theta(x)$ as calculated by each of the four methods, stating that there was a possibility of error. On comparing §§ 6 and 7 of that paper with the results of the present calculations I have found the following errata. In the two expressions for $\Theta(x)$ as definite integrals in § 6 (pp. 65, 66) at should be $\sqrt{a}t$, and in the third line of p. 66 the term $\frac{\sqrt{\pi}}{\sqrt{\mu}}$ is omitted from the right-hand member of the equation. When the corrections are made the results in §§ 6 and 7 are in complete accordance with those contained in the present paper.

November 28, 1887.

The following were elected Fellows:

C. Chree, M.A., King's College.
G. H. Bryan, B.A., Peterhouse.
J. R. Vaizey, B.A.

The following communications were made to the Society:

(1) *On the interaction of zinc and sulphuric acid.* By M. M. Pattison Muir, M.A., and R. H. Adie, B.A.

An experimental investigation of the nature of the products of the interaction of zinc, sulphuric acid, and water, and of the influence on the chemical change in question of temperature, changes in the relative masses of the reacting bodies, and the presence of traces of foreign metals in the zinc used.

(2) *On the Application of Lagrange's Equations to the Motion of a number of Cylinders in a Liquid when there is Circulation.* By A. B. Basset, M.A.

1. In a previous communication to the Society, I showed how Lagrange's equations might be employed to obtain the motion of a number of perforated solids in a liquid when there is circulation, and I proved that the modified Lagrangian function was

$$L = \mathfrak{T} + \Sigma(\mathfrak{X}u) - \mathfrak{K} + V \qquad (1),$$

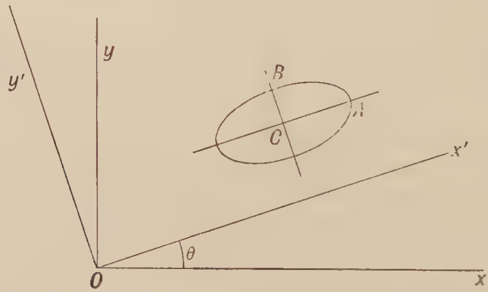
where $\mathfrak{T}$ is the kinetic energy due to the motion of the solids alone, u is any one of the component velocities of the solids, $\mathfrak{X}$ is the generalised component of momentum corresponding to u of the cyclic motion which remains after all the solids have been reduced to rest, $\mathfrak{K}$ is the kinetic energy due to the cyclic motion alone, and V is the potential of the impressed forces.

If we endeavour to calculate the right-hand side of (1), in the case of the two-dimensional motion of a number of cylinders in an infinite liquid when there is circulation round some or all the cylinders, it will be found that some of the terms become infinite. In order to avoid this difficulty, we must describe an imaginary fixed circular cylinder in the liquid, the radius of whose cross section is a very large quantity c, and then calculate the value of L for the space contained between the moving cylinders and the outer one, omitting all the terms which vanish when c becomes infinite. It will then be found on substituting the value of L thus obtained in Lagrange's equations and performing the differentiations, that all the terms which become infinite with c disappear, and we thus obtain the equations of motion of the cylinders.

2. The calculation of L can most easily be effected by employing the current function instead of the velocity potential, for the former function is always single valued unless any sources or sinks exist in the liquid.

Let u_1, v_1 be the component velocities of any cylinder S_1 along rectangular axes *fixed in the cylinder*, ω_1 its angular velocity, κ_1 the circulation round any closed circuit which embraces this cylinder once only.

Let the centre O of the cross section of the outer cylinder be the origin, and let x_1, y_1 be the co-ordinates of the centre of inertia of the cross section of S_1 referred to rectangular axes *fixed in space;* x_1', y_1' the co-ordinates of the same point referred to moving axes through O which are parallel to the directions of u_1, v_1. Also let χ be the current function and Ω be the velocity potential of the cyclic motion when all the cylinders are at rest.



In the figure let CA, CB be the axes of any one of the cylinders along which u_1, v_1 are measured. then

$$\mathfrak{X}_1 = \rho \iint \frac{d\chi}{dy'}\, dx'dy'$$

$$= -\rho \int \chi \frac{dx'}{ds}\, ds + \rho \left[\int \chi \frac{dx'}{ds}\, ds \right],$$

where the first integral is to be taken once round the circumference of the cross section of the outer cylinder, and the square brackets denote that the second integral is to be taken once round the circumferences of the cross sections of each of the moving cylinders.

At the surface of each of the moving cylinders χ is constant, hence the second integral vanishes, therefore

$$\mathfrak{X}_1 = -\rho \int \chi \frac{dx'}{ds}\, ds.$$

Let (r', θ') be polar co-ordinates of a point referred to Ox' as initial line, then at a sufficient distance from O, χ can be expanded in a series of the form

$$\chi = -m \log r' - \frac{1}{r'}(\mathfrak{A}_1 \cos\theta' + \mathfrak{B}_1 \sin\theta') + \dots$$

Therefore

$$\mathfrak{X}_1 = -\rho c \int_0^{2\pi} \left\{ m \log c + \frac{1}{c}(\mathfrak{A}_1 \cos\theta' + \mathfrak{B}_1 \sin\theta') + \dots \right\} \sin\theta' d\theta'$$

$$= -\pi\rho\mathfrak{B}_1 \quad \text{..(2).}$$

Similarly $$\mathfrak{Y}_1 = -\rho \iint \frac{d\chi}{dx'} dx'dy' = -\int \chi \frac{dy'}{ds} ds$$

$$= \pi\rho\mathfrak{A}_1 \quad \text{..................................(3).}$$

Again, if $\mathfrak{N}_1$ be the angular momentum about C of the cyclic motion,

$$\mathfrak{N}_1 = -\rho \iint \left\{ (x' - x_1') \frac{d\chi}{dx'} + (y' - y_1') \frac{d\chi}{dy'} \right\} dx'dy'$$

$$= -\rho \iint \left\{ x' \frac{d\chi}{dx'} + y' \frac{d\chi}{dy'} \right\} dx'dy' - \pi\rho (\mathfrak{A}_1 x_1' + \mathfrak{B}_1 y_1').$$

By Stokes' theorem the double integral

$$= -\tfrac{1}{2}\rho \int r^2 \frac{d\chi}{dn} ds + \tfrac{1}{2}\rho \left[\int r^2 \frac{d\chi}{dn} ds \right].$$

The first integral $= \pi\rho c^2 m$, the second integral may be written

$$-\tfrac{1}{2}\rho \left[\int r^2 d\Omega/ds \,.\, ds \right],$$

hence

$$\mathfrak{N}_1 = \pi\rho c^2 m - \tfrac{1}{2}\rho \left[\int r^2 \frac{d\Omega}{ds} ds \right] - \pi\rho (\mathfrak{A}_1 x_1' + \mathfrak{B}_1 y_1') \text{........(4).}$$

Also $$2\mathfrak{K} = \rho \int \chi \frac{d\chi}{dn} ds - \rho \left[\int \chi \frac{d\chi}{dn} ds \right]$$

$$= \rho c \int_0^{2\pi} \chi \frac{d\chi}{dr} d\theta + \rho \left[\int \chi \frac{d\Omega}{ds} ds \right]$$

$$= \rho c \int_0^{2\pi} \chi \frac{d\chi}{dr} d\theta + \rho \Sigma (\kappa\chi).$$

The integral

$$= \rho \int_0^{2\pi} \left\{ m \log c + \frac{1}{c} (\mathfrak{A} \cos \theta + \mathfrak{B} \sin \theta) \right\}$$

$$\times \left\{ m - \frac{1}{c} (\mathfrak{A} \cos \theta + \mathfrak{B} \sin \theta) \right\} d\theta = 2\pi\rho m^2 \log c.$$

Whence $$\mathfrak{K} = \pi\rho m^2 \log c + \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) \qquad \text{.....................(5).}$$

Hence we finally obtain

$$L = \mathfrak{T} + \pi\rho\Sigma\,(\mathfrak{A}v - \mathfrak{B}u) + \Sigma\,(\mathfrak{N}\omega)$$
$$- \pi\rho m^2 \log c - \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) + V \qquad \text{.........(6).}$$

If we substitute the preceding expression for L in Lagrange's equations and perform the differentiations, it will be found that the terms $\pi\rho c^2 m$ in $\mathfrak{N}$, and $\pi\rho m^2 \log c$ disappear; we may therefore write

$$L = \mathfrak{T} + \pi\rho\Sigma\,(\mathfrak{A}v - \mathfrak{B}u) + \Sigma\,(\mathfrak{N}\omega) - \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) + V \qquad \text{.........(7).}$$

$$\mathfrak{N} = -\tfrac{1}{2}\rho \left[\int r^2 \frac{d\Omega}{ds}\, ds \right] - \pi\rho\,(\mathfrak{A}x' + \mathfrak{B}y') \qquad \text{.........(8).}$$

The quantity $\mathfrak{T}$ which does not depend on the cyclic motion can be obtained by the ordinary methods. With respect to the other terms we must first obtain the values of χ and Ω; we must then draw from O a series of lines parallel to the directions of u_1, u_2 ..., and take each of these lines successively as the initial line, and expand χ in a series of the form

$$\chi = -m \log r - \frac{1}{r} (\mathfrak{A} \cos \theta + \mathfrak{B} \sin \theta) + \dots$$

which will determine the values of the $\mathfrak{A}$'s and $\mathfrak{B}$'s.

The velocities u, v and the co-ordinates x', y' expressed in terms of x, y, the co-ordinates of C referred to fixed axes, and the angle θ which CA makes with Ox, are given by the equations

$$\left.\begin{aligned} u &= \dot{x} \cos\theta + \dot{y} \sin\theta, \quad & v &= -\dot{x} \sin\theta + \dot{y} \cos\theta \\ x' &= x \cos\theta + y \sin\theta, \quad & y' &= -x \sin\theta + y \cos\theta \end{aligned}\right\} \text{.........(9).}$$

When there are several cylinders, the value of χ at the surfaces of the different cylinders is a function of their forms and positions, and is therefore a function of the co-ordinates; when there is only one cylinder the value of χ at its surface is an absolute constant.

We shall now give some examples.

A Circular Cylinder.

3. Let a be the radius, and σ the density of the cylinder, then

$$\mathfrak{T} = \tfrac{1}{2}\pi a^2 (\rho + \sigma)(\dot{x}^2 + \dot{y}^2),$$

$$\chi = -\frac{\kappa}{2\pi} \log \{(r' \cos \theta' - x)^2 + (r' \sin \theta' - y)^2\}^{\frac{1}{2}},$$

$$= -\frac{\kappa}{2\pi} \log r' + \frac{\kappa}{2\pi r'} (x \cos \theta' + y \sin \theta') + \&c.$$

Whence $\mathfrak{A} = -\kappa x/2\pi, \quad \mathfrak{B} = -\kappa y/2\pi.$

Taking for a moment the origin at the centre of the moving cylinder, the value of $\mathfrak{N}$ is

$$\mathfrak{N} = -\iint \frac{d\chi}{dr} r^2 dr\, d\theta = \frac{\kappa}{2\pi} \iint r\, dr\, d\theta = \tfrac{1}{2}\kappa (c^2 - a^2),$$

whence $\mathfrak{N}$ is constant, therefore disappears on differentiation.

The value of L may consequently be written

$$L = \tfrac{1}{2}\pi a^2 (\rho + \sigma)(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\kappa\rho (\dot{x}y - \dot{y}x),$$

and the equations of motion are

$$\left.\begin{aligned} \pi a^2 (\rho + \sigma)\ddot{x} + \kappa\rho\dot{y} &= X \\ \pi a^2 (\rho + \sigma)\ddot{y} - \kappa\rho\dot{x} &= Y \end{aligned}\right\} \quad \text{......................}(10),$$

which agree with the equations obtained by Prof. Greenhill*. If no impressed forces act, the cylinder, as was first shown by Lord Rayleigh†, will describe a circle in the same direction as that of the cyclic motion.

An Elliptic Cylinder.

4. If $\qquad x_1 - x' + \iota(y_1 - y') = a \cos(\xi - \iota\eta),$

where x_1, y_1 are current co-ordinates referred to Ox', Oy', then $\kappa\xi/2\pi$ and $-\kappa\eta/2\pi$ are the velocity potential and current function for cyclic motion round an elliptic cylinder.

Let $\qquad \lambda = (x_1 + \iota y_1)/a, \quad \mu = (x' + \iota y')/a,$

then since $\qquad \cos^{-1} x = -\iota \log (x + \sqrt{x^2 - 1}),$

we have $\quad \xi - \iota\eta = -\iota \log \{\lambda - \mu + \sqrt{(\lambda - \mu)^2 - 1}\}$

$$= -\iota \log 2\lambda - \iota \log \left(1 - \frac{\mu}{\lambda} + \ldots\right)$$

$$= -\iota (\log 2r_1 + \iota\theta_1) + \frac{\iota}{r_1} (x' + \iota y')(\cos \theta_1 - \iota \sin \theta_1).$$

* *Mess. Math.* Vol. IX. p. 115. † *Ibid.* Vol. VII. p. 14.

Therefore $$\eta = \log 2r_1 - \frac{1}{r_1}(x' \cos\theta_1 + y' \sin\theta_1),$$

whence $$\mathfrak{A} = -\kappa x'/2\pi, \quad \mathfrak{B} = -\kappa y'/2\pi.$$

Therefore $$\mathfrak{N} = \tfrac{1}{2}\kappa\rho(x^2+y^2) - \tfrac{1}{2}\rho\int r^2 \frac{d\Omega}{ds}\, ds.$$

Let R and Θ be polar co-ordinates of a point on the boundary referred to C as origin, and CA as initial line, and let $OCA = \epsilon$,

$$\begin{aligned} r^2 &= x^2 + y^2 + R^2 - 2R\sqrt{x^2+y^2}\cos(\epsilon+\Theta) \\ &= x^2 + y^2 + a^2(\cosh^2\eta\cos^2\xi + \sinh^2\eta\sin^2\xi) \\ &\quad - 2a\sqrt{x^2+y^2}(\cos\epsilon\cosh\eta\cos\xi - \sin\epsilon\sinh\eta\sin\xi), \end{aligned}$$

whence $$\tfrac{1}{2}\int r^2\frac{d\Omega}{ds}\,ds = \frac{\kappa\rho}{4\pi}\int_0^{2\pi} r^2 d\xi = \tfrac{1}{2}\kappa\rho(x^2+y^2) + \tfrac{1}{2}\kappa\rho a^2.$$

Hence $\mathfrak{N} = -\tfrac{1}{2}\kappa\rho a^2$, a constant.

The modified function may therefore be written

$$L = \mathfrak{T} + \tfrac{1}{2}\kappa\rho(y'u - x'v) = \tfrac{1}{2}(Pu^2 + Qv^2 + C\omega^2) + \tfrac{1}{2}\kappa\rho(\dot{x}y - \dot{y}x),$$

where P and Q are the effective inertias parallel to the major and minor axes of the cross section, and C is the effective moment of inertia about the axis of the cylinder.

The equations of motion are therefore

$$\left.\begin{aligned} &\frac{d}{dt}(Pu\cos\theta - Qv\sin\theta + \kappa\rho y) = X, \\ &\frac{d}{dt}(Pu\sin\theta + Qv\cos\theta - \kappa\rho x) = Y, \\ &C\frac{d\omega}{dt} - (P-Q)uv = N, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (11),$$

where X, Y are the components of the impressed forces parallel to the *fixed* axes, and N is the impressed couple about the axis of the cylinder.

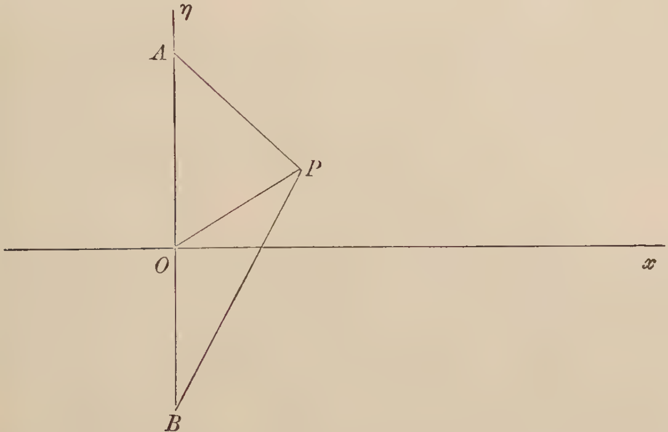
Two Cylinders.

5. Let us now consider the motion of two equal cylinders round which there is circulation in opposite directions, and which are initially projected with equal velocities parallel to Ox.

Let A and B be the common inverse points of the two cylinders, a the radius of either of them, u, v and u, $-v$ their velocities

parallel and perpendicular to Ox, y the ordinate of the centre of the cylinder A.

It is known from the theory of dipolar co-ordinates, that the cyclic motion is the same as would be produced by two rectilinear



vortices of circulations κ and $-\kappa$ situated at A and B, hence the value of χ will be

$$\chi = -\frac{\kappa}{2\pi}\log\frac{AP}{BP} = \frac{\kappa\eta}{2\pi}.$$

Also, if α be the value of η at the surface of the cylinder A, and $AB = 2c$,

$$a = c \operatorname{cosech} \alpha, \quad y = c \coth \alpha \;\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and

$$\Sigma(\kappa\chi) = \kappa^2\alpha/\pi.$$

Since this kind of cyclic motion could be produced by applying a uniform impulsive pressure $\kappa\rho$ to every point of that portion of AB which lies between the cylinders, we must have $\mathfrak{P} = 0$. Let (r, θ) be the co-ordinates of P referred to O, then

$$\chi = -\frac{\kappa}{4\pi}\log\frac{r^2 + c^2 - 2rc\sin\theta}{r^2 + c^2 + 2rc\sin\theta} = \frac{\kappa c}{\pi r}\sin\theta + \&c.,$$

whence

$$\mathfrak{A} = 0, \quad \mathfrak{B} = -\kappa c/\pi.$$

Therefore

$$L = \mathfrak{T} + 2\kappa c\rho u - \kappa\rho^2\alpha/2\pi.$$

Also if M be the mass per unit of length of either of the cylinders

$$\mathfrak{T} = (M + R)(u^2 + v^2).$$

The function R has been determined by Mr W. M. Hicks*, and also by Prof. Greenhill†, and its value is

$$R = \pi a^2 \rho \left\{1 + 2(1-q)^2 \Sigma_1^\infty \frac{q^m}{(1-q^{m+1})^2}\right\},$$

where $q = \epsilon^{-2\alpha}$.

If we suppose the cylinder B to be replaced by the fixed plane Ox which forms the boundary of the liquid, the value of L must be halved, and the equations of motion of the cylinder A will be

$$\frac{d}{dt}\left(\tfrac{1}{2}\frac{d\mathfrak{T}}{du} + \kappa\rho c\right) = X \quad \text{..................(13)},$$

$$\tfrac{1}{2}\frac{d}{dt}\frac{d\mathfrak{T}}{dv} - \tfrac{1}{2}\frac{d\mathfrak{T}}{dy} - \kappa\rho u\frac{dc}{dy} + \frac{\kappa^2}{4\pi}\frac{d\alpha}{dy} = Y \quad \text{............(14)}.$$

Now $\qquad c = \sqrt{y^2 - a^2}$ and $y = a\cosh\alpha$,

Therefore $\qquad \dfrac{dc}{dy} = \coth\alpha, \quad \dfrac{d\alpha}{dy} = \dfrac{1}{c},$

whence (14) becomes

$$\tfrac{1}{2}\frac{d}{dt}\frac{d\mathfrak{T}}{dv} - \tfrac{1}{2}\frac{d\mathfrak{T}}{dy} - \kappa\rho u\coth\alpha + \frac{\kappa^2}{4\pi c} = Y \quad \text{.........(15)}.$$

Let us now suppose that gravity is the only force in action, and that the plane boundary Ox is horizontal, forming, so to speak, the bed of the ocean; (13) and (15) respectively become

$$\left.\begin{aligned} Ru + \kappa\rho c &= \text{const.} = h \\ R\dot{v} + \tfrac{1}{2}(v^2 - u^2)\frac{dR}{dy} - \kappa\rho u\coth\alpha + \frac{\kappa^2\rho}{4\pi c} &= -(M - M')g \end{aligned}\right\} \text{...(15A)}.$$

These equations are satisfied by $v = 0$, u and y constant, provided u satisfies the quadratic

$$pu^2 - \kappa\rho u\coth\alpha + \frac{\kappa^2\rho}{4\pi c} + (M - M')g = 0 \quad \text{.........(16)},$$

where $p = -\tfrac{1}{2}dR/dy$. The roots of this quadratic will be real

provided $\qquad \kappa^2\rho^2\coth^2\alpha > p\left\{\dfrac{\kappa^2\rho}{\pi c} + 4(M - M')g\right\} \quad \text{............(17)}.$

* "On the Motion of Two Cylinders in a Fluid," *Quart. Journ.* Vol. XVI.
† "Functional Images in Cartesians," *Ibid.* Vol. XVIII. pp. 356—362.

CASE (i). Since p is positive the roots will always be real if

$$M' > M$$

and $$\kappa^2 \rho < \pi c (M' - M) g.$$

In this case the liquid is denser than the cylinder, and one of the roots of (16) will be positive and the other negative, and the positive root will be numerically greater than the negative root. Hence there will be two cases of steady motion, in one of which velocity of the cylinder will be in the *same* direction as that of the liquid, due to the circulation at points between the cylinder and plane; and in the other the velocity will be in the *opposite* direction; also the velocity in the former case will be greater than in the latter.

CASE (ii). $M' > M, \quad \kappa^2 \rho > 4\pi c (M' - M) g.$

In this case the roots of (16) will be both real and positive provided (17) is satisfied; hence the velocity in the two cases of steady motion will be in the *same* direction as that due to the circulation.

CASE (iii). $M > M'.$

In this case the cylinder is denser than the liquid, and the roots of (16), if real, must be both positive, hence the two velocities must be in the *same* direction as that due to the circulation.

CASE (iv). If either $g = 0$ or $M = M'$, (17) becomes

$$\pi \rho c \coth^2 \alpha > p.$$

Here both roots of (16) are positive, and the two velocities must be in the same direction as that due to the circulation.

This case has been discussed by Mr W. M. Hicks*.

CASE (v). Suppose that the cylinder is reduced to rest, and then let go. Since u and v are initially zero, the initial acceleration is

$$\dot{v} = -\frac{1}{4R\pi c}\{4\pi c (M - M') g + \kappa^2 \rho\} \quad \text{.........(18).}$$

Hence if the liquid is denser than the cylinder it is possible for the right-hand side to vanish; in which case the cylinder will remain in equilibrium under the combined action of gravity and the pressure due to the cyclic motion.

If the plane formed the upper boundary of the liquid the sign of g in these five cases would have to be reversed.

* *Quart. Journ.* Vol. XVII. p. 194.

The last three articles have been worked out for the sake of giving examples of the application of Lagrange's equations; but it would not be difficult to develope the subject of the last article considerably further. If, for example, we desired to investigate the motion when the circulations have different values κ, κ'; the current function due to the cyclic motion might be determined by writing down the current function due to two vortices of circulations κ, κ' situated at A and B respectively, and determining by means of dipolar co-ordinates another function ψ, such that the sum of all three functions is constant at the surfaces of the two cylinders.

Another problem of the same class is obtained by considering the conjugate functions,

$$\xi + \iota y = \tfrac{1}{2}\log\{(x + \iota y)^2 - c^2\}/c^2.$$

If α is positive the curve $\xi = -\alpha$ is a lemniscate consisting of two detached ovals. The current functions for the motion of cylinders whose cross sections are these curves have been given by myself*, but in applying these formulæ to two detached cylinders, it would be necessary to suppose them either rigidly connected together, or connected by a rod lying in the plane of the motion and passing through the foci, upon which the cylinders can slide.

(3) *Note on Kirchhoff's theory of the deformation of elastic plates.* By A. E. H. Love.

The theory here called Kirchhoff's is that worked out for Kirchhoff by his pupil Gehring, and will be found in No. XXX. of the *Vorlesungen über Mathematische Physik*, and in §§ 64 sq. of Clebsch's *Theorie der Elasticität fester Körper*. The object of this note is to call attention to some points in which the theory of the internal equilibrium of an element of the plate appears to be deficient in rigour, and at the same time to make it somewhat more lucid.

Objections have been raised to Kirchhoff's theory by M. Boussinesq on the grounds, (1) that it is obscure, (2) that it is founded on kinematical considerations which are only approximately true, and (3) that it is only capable of giving a first approximation. I shall shew that, by a necessary modification of the kinematical equations referred to, the equations derived therefrom can be made strictly accurate, and shall explain how the equations of equilibrium of an element may be made correct to any desired order of approximation. The criticisms of M.

* *Quart. Journ.* Vol. xx. pp. 242–246.

Boussinesq will be found in his memoir (*Etude sur l'equilibre et le mouvement des solides dont certaines dimensions sont très petites par rapport à d'autres*), published in Liouville's *Journal* in 1871, and they are substantially reproduced in de St Venant's translation of Clebsch's *treatise*, note on § 73.

Exactly similar considerations apply to the theory of wires. The result will be the establishment of Kirchhoff's views as to the kind of strain which can take place in an indefinitely thin wire or plate deformed in such a manner as to remain continuous.

It seems advisable to recapitulate the general method given by Kirchhoff, *Vorlesungen, XXVIII.*, for the treatment of elastic bodies some of whose dimensions are indefinitely small in comparison with others.

In this method we consider in the first place the equilibrium of an element of the body all whose dimensions are of the same order of linear magnitude as the indefinitely small dimension. When we know the potential energy due to the internal strain of such an element, we obtain by integration over the remaining dimensions the whole potential energy due to the elastic strain of the body. Then taking into account all the forces which act on the body we can form the equation of virtual work which will lead directly to the differential equations and boundary conditions of our problem.

Now let ϵ be a small quantity of the same order as that linear dimension of the body which is supposed small; and consider the equilibrium of a body all whose dimensions are of the order ϵ. Let (x, y, z) be rectangular coordinates of a point of this body and $F(x, y, z) = 0$ the equation of its surface; X, Y, Z the bodily forces per unit of mass applied to the solid, and $F, \underline{G}, H$ the surface-tractions. We write the six component stresses P, Q, R, S, T, U, viz.:

on the face yz,	P parallel to x,	U parallel to y,	T parallel to z,
on the face zx,	U parallel to x,	Q parallel to y,	S parallel to z,
on the face xy,	T parallel to x,	S parallel to y,	R parallel to z,

and the six strains e, f, g, a, b, c, viz.:

$$e = \frac{\partial u}{\partial x} \quad f = \frac{\partial v}{\partial y} \quad g = \frac{\partial w}{\partial z},$$

$$a = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad b = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad c = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

where u, v, w are the displacements, in the direction of the axes, of the particle originally at x, y, z. The stresses P, Q, R, S, T, U are linear functions of the strains e, f, g, a, b, c.

If l, m, n be the direction cosines of the outward-drawn normal to $F(x, y, z) = 0$ the surface tractions are

$$F = lP + mU + nT$$
$$G = lU + mQ + nS$$
$$H = lT + mS + nR.$$

Let us now take $x = \epsilon x'$, $y = \epsilon y'$, $z = \epsilon z'$ and substitute in the equation $F(x, y, z) = 0$, we obtain an equation which may be written $F'(x', y', z') = 0$ where F' contains only finite constants. Let

$$l' : m' : n' = \frac{\partial F'}{\partial x'} : \frac{\partial F'}{\partial y'} : \frac{\partial F'}{\partial z'} \text{ and } l'^2 + m'^2 + n'^2 = 1.$$

Write also
$$e' = \frac{\partial u}{\partial x'} \ldots a' = \frac{\partial w}{\partial y'} + \frac{\partial v}{\partial z'} \ldots,$$

and let $P' \ldots$ be the functions formed with the $e' \ldots$ in the same way as $P \ldots$ are formed with the $e \ldots$, the equations of equilibrium become three such as

$$\frac{\partial P'}{\partial x'} + \frac{\partial U'}{\partial y'} + \frac{\partial T'}{\partial z'} + \epsilon^2 \rho X = 0,$$

where ρ is the density; and at $F' = 0$ we have three surface conditions such as

$$l'P' + m'U' + n'T' = \epsilon F.$$

The solution for u, v, w consists of two parts, viz. that which corresponds to the bodily force when there is no surface-traction and that which corresponds to the surface-traction when there is no bodily force. Of these terms the former are of the order $\epsilon^2 \rho X$, and the latter of the order ϵF. Thus the displacements depending on the bodily forces are negligeable compared with those produced by the surface-tractions, and we may obtain the internal condition of the indefinitely small body by omitting the bodily forces.

Similar considerations show that if (x, y, z) always refer to points within an element of a body all whose dimensions are of the order ϵ, the strains $e \ldots$ are great compared with the displacements u, v, w.

By a thin plate is meant a mass of elastic material which in the natural state is bounded by two parallel planes and a cylindrical surface cutting them at right angles, the distance between the planes being very small compared with the least linear dimension of the space enclosed within the bounding curve on either of them.

The surface formed by all those particles of the plate which in the natural state lie on the plane midway between the two plane bounding surfaces is called the middle-surface of the plate.

Let any point on the middle-surface be taken as origin, and let α, β be rectangular coordinates of a point on the middle-surface. Let this surface be covered with a network of lines $\alpha = \text{const.}$, $\beta = \text{const.}$ at distances from each other comparable with the thickness of the plate. A prism whose normal section is one of these small rectangles will be called an "elementary prism" of the plate.

According to Kirchhoff's general method we have first to treat the equilibrium of one of these elementary prisms.

Let α, β be the coordinates of the centre P of one of these elementary prisms before strain. Imagine three line elements of the plate (1, 2, 3) to proceed from P, of which the first two are parallel to the axes of α, β and the third perpendicular to their plane before strain. Then after strain these axes will not be in general rectangular, but by means of them we can construct a system of rectangular axes whose origin is P, to which we can refer points in the prism whose centre is P, in the following way. The line-element 1 is to lie along the axis of x, and the plane of xy is to contain the line-elements 1 and 2. Then the line-element 2 will make an indefinitely small angle with the axis y, and the line-element 3 will make an indefinitely small angle with the axis z.

Let Q be a point in the prism whose centre is P; and before strain, let x, y, z be the coordinates of Q referred to the axes at P. Let ξ, η, ζ be the coordinates of P after strain referred to axes fixed in space and coinciding with the initial directions of α, β and the perpendicular to their plane. After strain let

$$x+u,\ y+v,\ z+w,$$

be the coordinates of Q referred to the (x, y, z) axes at P constructed as above described, and let the directions of these axes be connected with those of the fixed axes (ξ, η, ζ) by the scheme

	ξ	η	ζ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

.... (1).

then the coordinates of Q after strain referred to the fixed axes are

$$\left.\begin{aligned} &\xi + l_1(x+u) + l_2(y+v) + l_3(z+w), \\ &\eta + m_1(x+u) + m_2(y+v) + m_3(z+w), \\ &\zeta + n_1(x+u) + n_2(y+v) + n_3(z+w). \end{aligned}\right\} \text{.........(2).}$$

These are functions of the initial position of Q, i.e. of $\alpha + x$, $\beta + y$, and thus the differential coefficient of each of them with respect to α must be equal to that with respect to x, and so for β and y.

In forming these differential coefficients it is important to observe that u, v, w are not functions of α, β, for they are the relative displacements of a point Q of the prism whose centre is P, and the α, β are the same for all points of this prism, viz. they are the coordinates of P. The differential coefficients of any function with reference to α, β have reference to the difference of the values of the function at corresponding points of contiguous prisms, and u, v, w are functions defined with reference to the prism whose centre is P.

Kirchhoff's equations (*Vorlesungen*, s. 450) include the differential coefficients which in my notation would be $\frac{\partial u}{\partial \alpha}, \ldots$ and these are afterwards neglected as small in comparison with $\frac{\partial u}{\partial x}$, on the ground that $\frac{\partial u}{\partial \alpha}$ will be of the order u, which in the general theory of these bodies has been shewn to be negligible in comparison with $\frac{\partial u}{\partial x}$.

This is the point in which M. Boussinesq describes Kirchhoff's process as wanting in rigour. He explains the smallness of $\frac{\partial u}{\partial \alpha}, \ldots$ by saying that at similarly situated points of contiguous prisms the difference in the amount of the relative displacements is small compared with the difference in the amount of such displacements at near points in the same prism, and describes this as an assumption. We have just given reason for holding that the differential coefficients such as $\frac{\partial u}{\partial \alpha}$ do not exist, inasmuch as u, v, w have only been defined as functions of position in one elementary prism.

Omitting these terms the equations obtained are three of the type

$$l_1\left(1 + \frac{\partial u}{\partial x}\right) + l_2\frac{\partial v}{\partial x} + l_3\frac{\partial w}{\partial x} = \frac{\partial \xi}{\partial \alpha} + (x + u)\frac{\partial l_1}{\partial \alpha} + (y + v)\frac{\partial l_2}{\partial \alpha} + (z + w)\frac{\partial l_3}{\partial \alpha} \quad \ldots (3).$$

and three derived from these by putting y for x on the left-hand side and β for α on the right-hand side.

Introduce now three functions depending on the stretching of the middle surface, viz. the extensions of the line-elements 1, 2 and the sine of the angle which 2 makes with the axis y after strain. Call these σ_1, σ_2, ϖ, then

$$\left.\begin{array}{l} l_1(1+\sigma_1)=\dfrac{\partial\xi}{\partial x},\ \ m_1(1+\sigma_1)=\dfrac{\partial\eta}{\partial x},\ \ n_1(1+\sigma_1)=\dfrac{\partial\zeta}{\partial x} \\ (l_2+l_1\varpi)(1+\sigma_2)=\dfrac{\partial\xi}{\partial\beta},\ \ (m_2+m_1\varpi)(1+\sigma_2)=\dfrac{\partial\eta}{\partial\beta}, \\ \qquad\qquad\qquad\qquad (n_2+n_1\varpi)(1+\sigma_2)=\dfrac{\partial\zeta}{\partial x} \end{array}\right\}\ldots(4).$$

Also introduce six functions depending on the bending of the middle-surface, κ_1, λ_1, τ_1, κ_2, λ_2, τ_2 defined by the equations

$$\kappa_1 = l_3\frac{\partial l_2}{\partial\alpha}+m_3\frac{\partial m_2}{\partial\alpha}+n_3\frac{\partial n_2}{\partial\alpha}\ \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where λ_1, τ_1 are got from these by cyclical interchange of the suffixes 1, 2, 3 and κ_2, λ_2, τ_2, are the corresponding quantities with β in place of α.

In case the middle-surface be unextended, λ_1, κ_2 are the curvatures of lines through P initially parallel to α, β, and κ_1 depends on the angle one of the directions of principal curvature of the middle-surface after strain makes with the line initially parallel to α, so that $\kappa_2\lambda_1+\kappa_1^2=$ the measure of curvature.

Now multiply the set of equations of which (3) is the type by l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; and add each time and perform similar operations on the second set, we obtain six equations, viz.:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -\tau_1(y+v)+\lambda_1(z+w)+\sigma_1 \\ \frac{\partial v}{\partial x} &= -\kappa_1(z+w)+\tau_1(x+u) \\ \frac{\partial w}{\partial x} &= -\lambda_1(x+u)+\kappa_1(y+v), \\ \text{and}\qquad \frac{\partial u}{\partial y} &= -\tau_2(y+v)+\lambda_2(z+w)+\varpi \\ \frac{\partial v}{\partial y} &= -\kappa_2(z+w)+\tau_2(x+u)+\sigma_1 \\ \frac{\partial w}{\partial y} &= -\lambda_2(x+u)+\kappa_2(y+v) \end{aligned}\right\}\ldots\ldots\ldots(6).$$

According to the principles previously explained u, v, w are negligible compared with $\frac{\partial u}{\partial x}, \dots$ and thus these six equations reduce to

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -\tau_1 y + \lambda_1 z + \sigma_1 \\ \frac{\partial v}{\partial x} &= -\kappa_1 z + \tau_1 x \\ \frac{\partial w}{\partial x} &= -\lambda_1 x + \kappa_1 y, \\ \text{and} \qquad \frac{\partial u}{\partial y} &= -\tau_2 y + \lambda_2 z + \varpi \\ \frac{\partial v}{\partial y} &= -\kappa_2 z + \tau_2 x + \sigma_2 \\ \frac{\partial w}{\partial y} &= -\lambda_2 x + \kappa_2 y \end{aligned}\right\} \dots\dots\dots\dots\dots (7).$$

Since $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, and similarly for v and w, we deduce

$$\left.\begin{aligned} \tau_1 &= 0 \\ \tau_2 &= 0 \\ \kappa_2 + \lambda_1 &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (8).$$

Integrating equations (7) we obtain

$$\left.\begin{aligned} u &= u_0 - \kappa_1 yz + \lambda_1 zx + \sigma_1 x + \varpi y \\ v &= v_0 - \kappa_2 yz - \kappa_1 zx + \sigma_2 y \\ w &= w_0 - \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}\kappa_2 y^2 \end{aligned}\right\} \dots\dots\dots\dots (9),$$

where u_0, v_0, w_0 are functions of z, and we have put τ_1, $\tau_2 = 0$ and $\lambda_2 = -\kappa_1$.

The six strains are

$$\left.\begin{aligned} e &= \lambda_1 z + \sigma_1 \\ f &= -\kappa_2 z + \sigma_2 \\ g &= \frac{\partial w_0}{\partial z} \\ a &= \frac{\partial v_0}{\partial z} \\ b &= \frac{\partial u_0}{\partial z} \\ c &= -2\kappa_1 z + \varpi \end{aligned}\right\} \dots\dots\dots\dots\dots (10).$$

Thus the strains are independent of x, y. Hence so also are the stresses, and the stress equations reduce to

$$\left.\begin{array}{l} \dfrac{\partial T}{\partial z} = 0 \\ \dfrac{\partial S}{\partial z} = 0 \\ \dfrac{\partial R}{\partial z} = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

If we suppose that no tractions are applied to the bounding surfaces of the plate, these give us $S, T, R = 0$ throughout the element.

These three equations enable us to find u_0, v_0, w_0, and thus to express the strains in terms of σ_1, σ_2, ϖ, κ_2, λ_1, κ_1 and z, i.e. we can express them in terms of the deformation of the middle-surface and the distance from it.

The potential energy per unit volume due to the strain in the elementary prism can now be written down. If we integrate this along the line z, normal to the middle-surface, we obtain the potential energy per unit area of the piece of the middle-surface contained within the elementary prism. This is the same for all points of this piece of the middle-surface. If δA be the area of this piece and W the potential energy per unit area for any point of δA, then $\Sigma W \delta A$ over the middle-surface is the potential energy due to the strain in the plate. This, in the limit, is $\iint W d\alpha d\beta$, the integration extending over the middle-surface.

We have next to shew how the theory can be made to give equations which shall be accurate to any desired order of approximation. Returning to equations (6) we have in them to substitute the approximate values (9) found for u, v, w. Then apply the tests that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} \ldots$ and we shall on integrating deduce new values for u, v, w as functions of x, y. We shall then explain how to form the stress-equations, and find approximate values for S, T, R, and by utilizing the surface-values of these deduce expressions for u, v, w as functions of x, y, z. By repeating the process we can find expressions for the u, v, w which shall be true to any desired order of approximation.

The work is interesting as the tests $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}, \ldots$ shew that to the order of approximation to which we are working the measure of curvature, $-(\kappa_2 \lambda_1 + \kappa_1^2) = 0$, but they introduce no other fresh condition.

The equations must still give $\tau_1 = 0$, for the only terms in $\frac{\partial u}{\partial x}$ which are linear in y are $-\tau_1 y - \tau_1\sigma_1 y$, and there is no term in $\frac{\partial u}{\partial y}$ which is linear in x. In like manner, writing σ_2, v, τ_2 for σ_1, u, τ_1 and interchanging x, y we find $\tau_2 = 0$.

Again, the terms in $\frac{\partial w}{\partial x}$ which are linear in y are $\kappa_1 y + \kappa_1\sigma_2 y - \lambda_1 \varpi y$, and the terms in $\frac{\partial w}{\partial y}$ which are linear in x are $-\lambda_2 x - \lambda_2\sigma_1 x$. Hence we find $\kappa_1(1+\sigma_2) - \lambda_1\varpi = -\lambda_2(1+\sigma_1)$. Or since σ_1, σ_2, ϖ are all small we have $\lambda_2 = -\kappa_1(1+\sigma_2-\sigma_1)+\lambda_1\varpi$, which gives $\lambda_2 = -\kappa_1$ approximately as before, and also gives us a second approximation to the value of λ_2.

Using the second approximation to λ_2 in terms of the first order, and the first approximation in terms of the second order, we find

$$\frac{\partial u}{\partial x} = \lambda_1[z + w_0 - \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}x_2 y^2] + \sigma_1$$

$$\frac{\partial u}{\partial y} = -\kappa_1[z + w_0 - \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}\kappa_2 y^2] + \varpi$$
$$+ \{\kappa_1(\sigma_1 - \sigma_2) + \lambda_1\varpi\} z.$$

The condition $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial_2 u}{\partial y \partial x}$ gives

$$\lambda_1(\kappa_1 x + \kappa_2 y) = \kappa_1(\lambda_1 x - \kappa_1 y),$$

which requires
$$\kappa_2\lambda_1 + \kappa_1^2 = 0 \quad\text{.........................}\quad (12).$$

Hence,

$$u = u_0' + \lambda_1 zx - \kappa_1 yz + \sigma_1 x + \varpi y + [\kappa_1(\sigma_1 - \sigma_2) + \lambda_1\varpi] yz$$
$$+ w_0(\lambda_1 x - \kappa_1 y) - \tfrac{1}{6}\lambda_1^2 x^3 - \tfrac{1}{6}\kappa_2\kappa_1 y^3 \quad\text{......}(13),$$

where u_0' is an unknown function of z, w_0 is a known function of z which for an isotropic plate is

$$-\frac{m-n}{m+n}[\tfrac{1}{2}(\kappa_2 - \lambda_1)z^2 + (\sigma_1 + \sigma_2)z]^*,$$

and the term in x^2y, which has the coefficient $\frac{1}{2}(-\lambda_1\kappa_1 + \lambda_1\kappa_1)$, and that in y^2x, which has the coefficient $\frac{1}{2}(\kappa^2\lambda_1 + \kappa_1^2)$ have disappeared.

* The notation for the elastic constants is that of Thomson and Tait.

Again,

$$\frac{\partial v}{\partial x} = -\kappa_1 [z + w_0 - \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}\kappa_2 y^2],$$

$$\frac{\partial v}{\partial y} = -\kappa_2 [z + w_0 - \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}\kappa_2 y^2] + \sigma_2.$$

The condition $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$ gives

$$-\kappa_1 (\kappa_1 x + \kappa_2 y) = -\kappa_2 (-\lambda_1 x + \kappa_1 y),$$

so that $\kappa_2 \lambda_1 + \kappa_1^2 = 0$ as before.

Hence

$$v = v_0' - \kappa_1 zx - \kappa_2 yz + \sigma_2 y - w_0 (\kappa_1 x + \kappa_2 y) + \tfrac{1}{6}\lambda_1 \kappa_1 x^3 - \tfrac{1}{6}\kappa_2^2 y^3 \ldots (14),$$

where v_0' is a function of z.

If u_0 and v_0 denote now known functions of z $\left(\text{determined by the equations } S,\ T,\ R = 0 \text{ and } a = \dfrac{\partial v_0}{\partial z},\ b = \dfrac{\partial u_0}{\partial z}\right)$, we shall have

$$\frac{\partial w}{\partial x} = -\lambda_1 (x + u_0 - \kappa_1 yz + \lambda_1 zx + \sigma_1 x + \varpi y) + \kappa_1 (y + v_0 - \kappa_2 yz - \kappa_1 zx + \sigma_2 y),$$

$$\frac{\partial w}{\partial y} = \kappa_1 (x + u_0 - \kappa_1 yz + \lambda_1 zx + \sigma_1 x + \varpi y) + \kappa_2 (y + v_0 - \kappa_2 yz - \kappa_1 zx + \sigma_2 y) - [\kappa_1 (\sigma_1 - \sigma_2) + \lambda_1 \varpi]\, x.$$

The condition $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial^2 w}{\partial y \partial x}$ is identically satisfied, and we obtain

$$\begin{aligned} w = w_0' &- \tfrac{1}{2}\lambda_1 x^2 + \kappa_1 xy + \tfrac{1}{2}\kappa_2 y^2 - \kappa_1 (\kappa_2 - \lambda_1)\, xyz \\ &- \tfrac{1}{2}(\lambda_1^2 + \kappa_1^2)\, zx^2 - \tfrac{1}{2}(\kappa_2^2 + \kappa_1^2)\, zy^2 - \tfrac{1}{2}\sigma_1 \lambda_1 x^2 + \tfrac{1}{2}\sigma_2 \kappa_2 y^2 - \lambda_1 \varpi xy \\ &- (\lambda_1 u_0 - \kappa_1 v_0)\, x + (\kappa_1 u_0 + \kappa_2 v_0)\, y \ldots\ldots (15), \end{aligned}$$

where w_0' is a function of z.

We find for the six strains

$$e = \lambda_1 z + \sigma_1 + w_0 \lambda_1 - \tfrac{1}{2}\lambda_1^2 x^2.$$

$$f = -\kappa_2 z + \sigma_2 - w_0 \kappa_2 - \tfrac{1}{2}\kappa_2 y^2.$$

$$g = \frac{\partial w_0'}{\partial z} - x\left(\lambda_1 \frac{\partial u_0}{\partial z} - \kappa_1 \frac{\partial v_0}{\partial z}\right) + y\left(\kappa_1 \frac{\partial u_0}{\partial z} + \kappa_2 \frac{\partial v_0}{\partial z}\right) - \kappa_1 (\kappa_2 - \lambda_1)\, xy - \tfrac{1}{2}(\lambda_1^2 + \kappa_1^2)\, x^2 - \tfrac{1}{2}(\kappa_1^2 + \kappa_2^2)\, y^2.$$

$$a = \frac{\partial v_0'}{\partial z} - \frac{\partial w_0}{\partial z}(\kappa_1 x + \kappa_2 y) + (\kappa_1 u_0 + \kappa_2 v_0) - \kappa_1(\kappa_2 - \lambda_1)zx$$
$$- (\kappa_2^2 + \kappa_1^2)yz + \sigma_2\kappa_2 y - \lambda_1 \varpi x.$$

$$b = \frac{\partial u_0'}{\partial z} + \frac{\partial w_0}{\partial z}(\lambda_1 x - \kappa_1 y) - (\lambda_1 u_0 - \kappa_1 v_0) - \kappa_1(\kappa_2 - \lambda_1)yz$$
$$- (\lambda_1^2 + \kappa_1^2)zx + (\sigma_1 - \sigma_2)\kappa_1 y - \sigma_1\lambda_1 x.$$

$$c = -2\kappa_1 z + \varpi - 2w_0\kappa_1 + [\kappa_1(\sigma_1 - \sigma_2) + \lambda_1\varpi]z - 2\kappa_1(\kappa_2 y^2 - \lambda_1 x^2).$$

The functions u_0', v_0', w_0' are to be determined from the stress-equations.

In the general case of an æolotropic solid the stresses are linear functions of the strains, and the equations contain differential coefficients of e, f, g, a, b, c with respect to x, y, z, these are linear functions of x, y, z. The equations also contain the differential coefficients $\frac{\partial^2 u_0'}{\partial z^2}, \frac{\partial^2 v_0'}{\partial z^2}, \frac{\partial^2 w_0'}{\partial z^2}$. On integrating we should find general expressions for u_0', v_0', w_0'. It is important to observe that for the purpose of the approximation we may finally put $x, y = 0$ as it is only necessary to know the strain at all points of a line initially normal to the middle surface. The potential energy will be found by first integrating along one of these lines and then for all the lines.

In the general case u_0, v_0 are not zero, and the best way is to calculate $\frac{\partial P}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial T}{\partial x}, \frac{\partial S}{\partial y}$, then substituting we find equations for $\frac{\partial T}{\partial z}, \frac{\partial S}{\partial z}, \frac{\partial R}{\partial z}$. On integrating these we may introduce the surface-values of S, T, R. If the solid be isotropic the work is much simpler. In this case $u_0, v_0 = 0$ and

$$w_0 = -\frac{m-n}{m+n}\left[(\kappa_2 - \lambda_1)\frac{z^2}{2} + (\sigma_1 + \sigma_2)z\right].$$

We find

$$\frac{\partial T}{\partial x} = n\frac{\partial b}{\partial x} = n\left[\lambda_1\frac{\partial w_0}{\partial z} - (\kappa_1^2 + \lambda_1^2)z - \sigma_1\lambda_1\right].$$

$$\frac{\partial S}{\partial y} = n\frac{\partial a}{\partial y} = n\left[-\kappa_2\frac{\partial w_0}{\partial z} - (\kappa_2^2 + \kappa_1^2)z + \kappa_2\sigma_2\right].$$

Thus the stress equation involving R becomes

$$\frac{\partial R}{\partial z} + \rho Z + n\left[(\kappa_2 - \lambda_1)\frac{m-n}{m+n}\{(\kappa_2 - \lambda_1)z + (\sigma_1 + \sigma_2)\}\right.$$
$$\left. - (\kappa_2^2 + \lambda_1^2 + 2\kappa_1^2)z + \kappa_2\sigma_2 - \lambda_1\sigma_1\right] = 0 \ldots\ldots (16).$$

On integrating this we obtain two equations for the $\kappa_2, \lambda_1, \kappa_1, \sigma_1, \sigma_2$ as functions of the surface-values of R, and these equations involve one unknown constant, thus they are equivalent to one relation between these quantities. Determining the constant from one surface-equation and substituting in the equation

$$R = (m-n)(e+f+g) + 2ng,$$

we should find $\frac{\partial w_0'}{\partial z}$.

We find that R is a quantity of the order $(\kappa_2 - \lambda_1)^2 \frac{\epsilon^2}{2}$, and $\lambda_1 \sigma_1 \epsilon$ where ϵ is of the order of the thickness, and the line-integral along the normal to the middle-surface of the bodily force is a quantity of the same order. The way in which the bodily force occurs in the equations of equilibrium is that the line integrals of the X, Y, Z and of the Xz, Yz enter, the form of the equation just written down shews that the effect produced by the surface-forces R is the same as if their resultant were applied directly to the middle-surface. This is a result otherwise obtained by M. Boussinesq.

For an isotropic plate we should still find $\frac{\partial v_0'}{\partial z}, \frac{\partial u_0'}{\partial z} = 0$, so that this order of approximation will not enable us to include the effect of the forces S, T.

The method we have adopted is capable of being extended so as to give any desired order of approximation, but it is much simpler to form the equation of virtual work directly from the strains as given by equations (10) as was done by Kirchhoff. I have given this second approximation partly for the sake of the manner in which it appears that the measure of curvature is very small, and partly because it serves to take away the reproach of M. Boussinesq, that the stresses S T, R cannot be found by Kirchhoff's method. The above work would be different for an æolotropic plate, and would, generally speaking, yield the values of S, T as well as that of R. It will be readily seen that it is quite unnecessary to know the values of S, T, R for the purpose of finding the form assumed by the plate under given forces, though it may be interesting if we wish to discover whether they can ever be important enough to produce a sensible tendency to rupture. The result is that they are always small quantities of the order of the square of the thickness.

(4) *A new Geometrical Interpretation of the Quaternion Analysis.* By J. BRILL, M.A.

1. In the September number of the *Messenger of Mathematics* I described a new method for the geometrical representation of complex quantities in which the complex quantities were represented by lines instead of points, tangential co-ordinates being substituted for cartesians. I now propose to apply the same ideas to the quaternion analysis, and thus to obtain a three-dimensional analogue of my former method. As in the former case, and for the same reason, it will be found that the analogy of the new interpretation to the old is not quite complete.

2. In the paper alluded to in the preceding paragraph, in order to lead up to the question of the addition of lines, I introduced the idea of the mean line. It will be found that the theorems connected with this line have their analogues in geometry of three dimensions, and that these furnish us with material for dealing with the question of the addition of planes. Thus we have the following theorem :

O is a fixed point, and through O a straight line is drawn meeting n fixed planes in the points $r_1, r_2, \ldots\ldots, r_n$. A point R is taken on this straight line so that

$$\frac{m_1+m_2+\ldots\ldots+m_n}{OR}=\frac{m_1}{Or_1}+\frac{m_2}{Or_2}+\ldots\ldots+\frac{m_n}{Or_n}.$$

The locus of R will be a plane.

To prove this take a system of three rectangular axes through O, and let the equations of the n planes referred to this system be

$$\begin{aligned}
&u_1x+v_1y+w_1z-1=0,\\
&u_2x+v_2y+w_2z-1=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&u_nx+v_ny+w_nz-1=0.
\end{aligned}$$

Then, if α, β, γ be the angles made with the axes by the line through O, we have

$$\begin{aligned}
\frac{1}{Or_1}&=u_1\cos\alpha+v_1\cos\beta+w_1\cos\gamma,\\
\frac{1}{Or_2}&=u_2\cos\alpha+v_2\cos\beta+w_2\cos\gamma,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{1}{Or_n}&=u_n\cos\alpha+v_n\cos\beta+w_n\cos\gamma.
\end{aligned}$$

Thus the equation of the locus of R is

$$\frac{m_1 + m_2 + \ldots\ldots + m_n}{r} = m_1 (u_1 \cos\alpha + v_1 \cos\beta + w_1 \cos\gamma) + \ldots\ldots$$
$$+ m_n (u_n \cos\alpha + v_n \cos\beta + w_n \cos\gamma);$$

i.e.
$$x \frac{\Sigma mu}{\Sigma m} + y \frac{\Sigma mv}{\Sigma m} + z \frac{\Sigma mw}{\Sigma m} - 1 = 0.$$

We shall speak of this as the mean plane, with respect to O, of the n given planes for the multiples $m_1, m_2, \ldots\ldots, m_n$ respectively.

3. Let the equation of a plane be given in the form

$$ux + vy + wz - 1 = 0,$$

where u, v, w are the reciprocals of the intercepts on the axes of co-ordinates. We shall denote the position of this plane by the expression $iu + jv + kw$, where i, j, k are symbols obeying the laws $i^2 = j^2 = k^2 = ijk = -1$. Sometimes we shall find it convenient to replace the expression $iu + jv + kw$ by the single symbol s.

It is clear that any plane parallel to the given one will be represented by a scalar multiple of the expression which denotes the given plane. The plane at infinity will be represented by zero, and any plane through the origin will be represented by an infinite vector. We shall still, for the sake of convenience, continue to speak of an expression of the form $iu + jv + kw$ as a vector although it is no longer taken to represent the relative position of two points.

4. Let $u_1x + v_1y + w_1z - 1 = 0$ and $u_2x + v_2y + w_2z - 1 = 0$ be the equations of two planes. Consider the plane

$$m(u_1x + v_1y + w_1z - 1) + n(u_2x + v_2y + w_2z - 1) = 0,$$

or as it may be written

$$\frac{mu_1 + nu_2}{m+n} x + \frac{mv_1 + nv_2}{m+n} y + \frac{mw_1 + nw_2}{m+n} z - 1 = 0.$$

Let this be equivalent to $ux + vy + wz - 1 = 0$, then we have

$$(m+n)\,u = mu_1 + nu_2,$$
$$(m+n)\,v = mv_1 + nv_2,$$
$$(m+n)\,w = mw_1 + nw_2.$$

Thus if we write

$$s = iu + jv + kw, \quad s_1 = iu_1 + jv_1 + kw_1, \quad s_2 = iu_2 + jv_2 + kw_2,$$

then we have $\qquad (m+n)\,s = ms_1 + ns_2.$

The plane s is what we have called the mean plane, with respect to the origin, of the planes s_1 and s_2 for multiples m and n respectively. If s be the mean of the planes $s_1, s_2, \ldots\ldots, s_n$ for multiples $m_1, m_2, \ldots\ldots, m_n$ respectively, then we have

$$(m_1 + m_2 + \ldots\ldots + m_n)\, s = m_1 s_1 + m_2 s_2 + \ldots\ldots + m_n s_n.$$

If we make $m_1 = m_2 = \ldots\ldots = m_n$, then the above equation becomes $ns = s_1 + s_2 + \ldots\ldots + s_n$, and s coincides with the polar plane of the origin with respect to the n given planes. Thus, to obtain the plane which is the sum of n given planes, we have to draw the polar plane of the origin with respect to these planes and then to draw a plane parallel to this at one-nth the distance from the origin.

If we have $m_1 + m_2 + \ldots\ldots + m_n = 0$ then s will denote some plane through the origin, and it will be seen that cases such as $s_1 - s_2$, $s_1 + s_2 - 2s_3$, &c. will need a separate investigation. We will, however, only trouble ourselves with the case $s_1 - s_2$ as all the rest may be reduced to this. Thus, if s_4 be the polar plane of the origin with respect to s_1 and s_2, we have $s_1 + s_2 = 2s_4$ and therefore

$$s_1 + s_2 - 2s_3 = 2\,(s_4 - s_3).$$

The equation of the plane $s_1 - s_2$ is

$$(u_1 - u_2)\, x + (v_1 - v_2)\, y + (w_1 - w_2)\, z - 1 = 0.$$

This is obviously parallel to

$$(u_1 - u_2)\, x + (v_1 - v_2)\, y + (w_1 - w_2)\, z = 0,$$

the plane which contains the origin and the line of intersection of s_1 and s_2. Further, the equation of $s_1 - s_2$ may be written in the form

$$u_1 x + v_1 y + w_1 z - 1 - (u_2 x + v_2 y + w_2 z) = 0,$$

which shews that it passes through the intersection of s_1 with a plane drawn through the origin parallel to s_2. Thus, to construct the plane $s_1 - s_2$, we draw through the origin a plane parallel to s_2, and through the intersection of this with s_1 we draw a plane parallel to that which contains the origin and the line of intersection of s_1 and s_2.

5. Let p be the perpendicular from the origin on the plane

$$ux + vy + wz - 1 = 0,$$

and let α, β, γ be the angles that it makes with the axes of coordinates. Also, let a, b, c be the intercepts of the given plane upon the axes. Then

$$p = a \cos \alpha = b \cos \beta = c \cos \gamma,$$

and therefore

$$iu + jv + kw = i\frac{1}{a} + j\frac{1}{b} + k\frac{1}{c} = \frac{1}{p}(i \cos \alpha + j \cos \beta + k \cos \gamma).$$

We are now in a position, with the aid of this formula, to discuss the question of the multiplication of two or more planes. Thus, if s_1 and s_2 denote two planes, we have

$$s_1s_2 = \frac{1}{p_1p_2}(i\cos\alpha_1 + j\cos\beta_1 + k\cos\gamma_1)(i\cos\alpha_2 + j\cos\beta_2 + k\cos\gamma_2)$$

$$= \frac{1}{p_1p_2}\{-(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2)$$
$$+ i(\cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1) + j(\cos\gamma_1\cos\alpha_2 - \cos\gamma_2\cos\alpha_1)$$
$$+ k(\cos\alpha_1\cos\beta_2 - \cos\alpha_2\cos\beta_1)\}.$$

Let θ be that dihedral angle between the two planes within which the origin lies, then

$$S\,.\,s_1s_2 = \frac{\cos\theta}{p_1p_2}, \text{ and } TV(s_1s_2) = \frac{\sin\theta}{p_1p_2}.$$

The vector portion of s_1s_2 is a plane, and, if we write $s/c = V\,.\,s_1s_2$, we have $p = p_1p_2/c\sin\theta$. Further, the direction cosines of the perpendicular from the origin on s are proportional to

$$\cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1,$$
$$\cos\gamma_1\cos\alpha_2 - \cos\gamma_2\cos\alpha_1,$$
$$\cos\alpha_1\cos\beta_2 - \cos\alpha_2\cos\beta_1.$$

Thus the plane s is perpendicular to the line of intersection of s_1 and s_2, and its distance from the origin is $p_1p_2/c\sin\theta$.

Since $V\,.\,s_1s_2 = -V\,.\,s_2s_1$, we see that $V\,.\,s_1s_2$ and $V\,.\,s_2s_1$ denote a pair of parallel planes at equal distances from the origin on opposite sides of it, their orientation and the distance of each from the origin being given by the preceding paragraph. It only remains to determine which is which. To specify this, let O be the origin, OL the perpendicular from it on s_1, and OM that on s_2. The perpendiculars from O on $V\,.\,s_1s_2$ and $V\,.\,s_2s_1$ will have to be measured off on a line drawn through O perpendicular to OL and OM, the perpendicular on $V\,.\,s_1s_2$ being measured in such a sense that a right-handed screw motion about it would tend to turn OL towards OM. The perpendicular on $V\,.\,s_2s_1$ would have to be measured in the opposite sense, *i.e.* so that a right-handed screw motion about it would tend to turn OM towards OL*.

If the two planes be at right angles to each other we have $S\,.\,s_1s_2 = 0$. Thus the product of two planes at right angles to one another is a plane at right angles to both of them, and, if

* I have taken the fundamental system of axes to be a right-handed screw system.

this plane be denoted by s/c, its distance from the origin will be p_1p_2/c.

If the two planes be parallel then we have

$$V \, . \, s_1 s_2 = - V \, . \, s_2 s_1 = 0\,;$$

and thus the two products s_1s_2 and s_2s_1 will be entirely scalar, and each of them will be equal to $-1/p_1p_2$. Further, if we make $s_2 = s_1$, we have $Ts_1 = 1/p_1$. Thus we see that the equation $Ts = 1/a$ may be taken as the tangential equation of a sphere of radius a having its centre at the origin.

6. Before proceeding to discuss the multiplication of three planes we will seek to obtain the interpretation of some quaternion formulæ with the aid of the preceding article.

If $Ts = Tt$, then we have $T(ms + nt) = T(ns + mt)$, and from this we easily deduce the following theorem:

Let (A) and (B) be two tangent planes of a sphere having the point O for its centre; and let (X) and (Y) be the mean planes, with respect to O, of (A) and (B) for multiples m, n and n, m respectively. Then (X) and (Y) are tangent planes of a sphere concentric with the former.

7. If $Ts = Tt$, then we have $S(s-t)(s+t) = 0$, from which we deduce the following:

Let (A) and (B) be two tangent planes of a sphere having its centre at the point O. Through O draw a plane parallel to (B), and through the intersection of this with (A) draw a plane parallel to that containing O and the line of intersection of (A) and (B). This last plane will be perpendicular to the polar plane of O with respect to (A) and (B).

8. We will next consider the equation $V(s-t)(s+t) = 2Vst$.

Let O be a given point, and let p_1 and p_2 be the perpendiculars from it on two given planes (A) and (B). Through O draw a plane parallel to (B), and through the intersection of this with (A) draw a plane (X) parallel to that containing O and the line of intersection of (A) and (B). Then, if (Y) be the polar plane of O with respect to (A) and (B), the above equation becomes

$$V \, . \, (X)(Y) = V \, . \, (A)(B).$$

Thus, if q_1 and q_2 be the perpendiculars from O on (X) and (Y), we have

$$\sin(A, B) : \sin(X, Y) :: p_1p_2 : q_1q_2,$$

where (A, B) and (X, Y) denote the dihedral angles between (A) and (B), and between (X) and (Y), respectively.

9. An interesting property of the mean plane follows from the equation

$$T^2(ms+nt) = m^2 \,.\, T^2 s + n^2 \,.\, T^2 t - 2mnSst.$$

Let (X) be the mean plane, with respect to O, of the given planes (A) and (B) for multiples m and n respectively; and let OL, OM, ON be the perpendiculars from O on (X), (A), (B) respectively. Then we have

$$\frac{(m+n)^2}{OL^2} = \frac{m^2}{OM^2} + \frac{n^2}{ON^2} - 2\frac{mn}{OM\,.\,ON}\cos(A, B),$$

where (A, B) is that dihedral angle between the two given planes within which O lies.

It will be easily seen that this theorem admits of extension. In fact, if (X) be the mean plane, with respect to O, of the n planes $(A_1), (A_2), \ldots\ldots, (A_n)$ for multiples $m_1, m_2, \ldots\ldots, m_n$ respectively; and if $OL, OM_1, OM_2, \ldots\ldots, OM_n$ be the perpendiculars from O on $(X), (A_1), (A_2), \ldots\ldots, (A_n)$ respectively, then we have

$$\frac{(m_1+m_2+\ldots\ldots+m_n)^2}{OL^2} = \Sigma\frac{m_1^2}{OM_1^2} - 2\Sigma\frac{m_1 m_2}{OM_1\,.\,OM_2}\cos(A_1, A_2).$$

Another interesting property of the mean plane, closely connected with this, follows from the equation

$$\begin{aligned} T^2(m_1 s_1 + m_2 s_2 + \ldots\ldots + m_n s_n) + m_1 S\,.\,s_1(m_1 s_1 + m_2 s_2 + \ldots\ldots + m_n s_n) \\ + m_2 S\,.\,s_2(m_1 s_1 + m_2 s_2 + \ldots\ldots + m_n s_n) + \&c. = 0. \end{aligned}$$

From L draw $LN_1, LN_2, \ldots\ldots, LN_n$ respectively perpendicular to $OM_1, OM_2, \ldots\ldots, OM_n$. Then we have, by aid of the above equation,

$$m_1\frac{ON_1}{OM_1} + m_2\frac{ON_2}{OM_2} + \ldots\ldots + m_n\frac{ON_n}{OM_n} = m_1 + m_2 + \ldots\ldots + m_n,$$

or as it may be written

$$m_1\frac{M_1N_1}{OM_1} + m_2\frac{M_2N_2}{OM_2} + \ldots\ldots + m_n\frac{M_nN_n}{OM_n} = 0.$$

10. Suppose that we have three planes s_1, s_2, s_3, and let PA, PB, PC be the respective intersections of s_2 and s_3, of s_3 and s_1, and of s_1 and s_2. Draw a sphere of unit radius, having its centre at P, and let it cut PA, PB, PC at A, B, C respectively. We have the theorem

$$S\,.\,Vs_1s_2Vs_2s_3 = s_2^2 Ss_3s_1 - Ss_1s_2Ss_2s_3,$$

and from this we will deduce a theorem concerning the spherical triangle ABC. Now the perpendiculars from the origin on $V.\,s_1s_2$ and $V.\,s_2s_3$ are respectively parallel to PC and PA, and we have

also $TV(s_1s_2) = \sin C/p_1p_2$ and $TV(s_2s_3) = \sin A/p_2p_3$, and consequently $S.Vs_1s_2Vs_2s_3 = -\sin C \sin A \cos b/p_1p_2^2p_3$. Thus we have

$$\sin C \sin A \cos b = \cos B + \cos C \cos A.$$

Comparing this with the formula that can be deduced from the same analysis by means of Sir W. Hamilton's method of interpretation*, we see that each can be deduced from the other by means of the theory of the polar triangle.

11. We have the equation†

$$V.s_1Vs_2s_3 = s_3Ss_1s_2 - s_2Ss_3s_1,$$

and squaring this we obtain

$$-(TV.s_1Vs_2s_3)^2 = s_3^2(Ss_1s_2)^2 + s_2^2(Ss_3s_1)^2 - 2Ss_2s_3Ss_3s_1Ss_1s_2.$$

Now the perpendicular from the origin on $V.s_2s_3$ is parallel to PA, and $TV(s_2s_3) = \sin A/p_2p_3$; therefore

$$TV.s_1Vs_2s_3 = \sin A \sin\theta/p_1p_2p_3,$$

where θ is the angle between PA and the normal to s_1. Now, if p_a be the arc of a great circle drawn through A perpendicular to the side BC of the spherical triangle ABC, we have $\sin\theta = \cos p_a$. Thus we deduce the formula

$$\sin^2 A . \cos^2 p_a = 2\cos A \cos B \cos C + \cos^2 C + \cos^2 B.$$

This formula may be written in the form

$$\sin^2 A . \sin^2 p_a = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C.$$

12. Sir W. Hamilton gives the following equation‡

$$\begin{aligned} &s_1^2s_2^2s_3^2s_4^2 + (Ss_2s_3Ss_1s_4)^2 + (Ss_3s_1Ss_2s_4)^2 + (Ss_1s_2Ss_3s_4)^2 \\ &\quad + 2s_1^2Ss_2s_3Ss_2s_4Ss_3s_4 + 2s_2^2Ss_3s_1Ss_3s_4Ss_1s_4 + 2s_3^2Ss_1s_2Ss_1s_4Ss_2s_4 \\ &\quad + 2s_4^2Ss_1s_2Ss_2s_3Ss_3s_1 = 2Ss_3s_1Ss_1s_2Ss_2s_4Ss_3s_4 + 2Ss_1s_2Ss_2s_3Ss_3s_4Ss_1s_4 \\ &\quad + 2Ss_2s_3Ss_3s_1Ss_1s_4Ss_2s_4 + s_2^2s_3^2(Ss_1s_4)^2 + s_3^2s_1^2(Ss_2s_4)^2 + s_1^2s_2^2(Ss_3s_4)^2 \\ &\quad + s_1^2s_4^2(Ss_2s_3)^2 + s_2^2s_4^2(Ss_3s_1)^2 + s_3^2s_4^2(Ss_1s_2)^2. \end{aligned}$$

Suppose that the four planes s_1, s_2, s_3, s_4 form a tetrahedron $ABCD$, enclosing the origin, A, B, C, D being the vertices respectively opposite to s_1, s_2, s_3, s_4. We shall represent the dihedral angle contained by the faces respectively opposite A and B by the symbol (A, B), and similarly for the other dihedral angles, it being understood that the internal dihedral angles of the tetrahedron are meant. Then the above formula becomes

* Tait's *Quaternions*, p. 56.
† Hamilton's *Elements*, p. 316; Tait's *Quaternions*, p. 44.
‡ Hamilton's *Elements*, p. 347.

$$\begin{aligned}1 &+ \cos^2(B,C)\cos^2(A,C) + \cos^2(C,A)\cos^2(B,D) + \cos^2(A,B)\cos^2(C,D)\\ &- 2\cos(B,C)\cos(B,D)\cos(C,D) - 2\cos(C,A)\cos(C,D)\cos(A,D)\\ &- 2\cos(A,B)\cos(A,D)\cos(B,D) - 2\cos(A,B)\cos(B,C)\cos(C,A)\\ &= 2\cos(C,A)\cos(A,B)\cos(B,D)\cos(C,D)\\ &+ 2\cos(A,B)\cos(B,C)\cos(C,D)\cos(A,D)\\ &+ 2\cos(B,C)\cos(C,A)\cos(A,D)\cos(B,D)\\ &+ \cos^2(A,D) + \cos^2(B,D) + \cos^2(C,D)\\ &+ \cos^2(B,C) + \cos^2(C,A) + \cos^2(A,B).\end{aligned}$$

13. We will next consider the equation

$$S.s_1(s_2-s_3) + S.s_2(s_3-s_1) + S.s_3(s_1-s_2) = 0.$$

Suppose that we have three planes (A), (B), (C). We can draw, by means of the construction given in Art. 4, three other planes (X), (Y), (Z) such that

$$(X) = (B) - (C), \quad (Y) = (C) - (A), \quad (Z) = (A) - (B);$$

and we have

$$S.(A)(X) + S.(B)(Y) + S.(C)(Z) = 0.$$

Hence if $p_1, p_2, p_3, q_1, q_2, q_3$ be the perpendiculars from the origin on (A), (B), (C), (X), (Y), (Z) respectively, we have

$$\frac{\cos(A,X)}{p_1q_1} + \frac{\cos(B,Y)}{p_2q_2} + \frac{\cos(C,Z)}{p_3q_3} = 0,$$

it being understood that by the dihedral angle between any two of the planes that particular angle is intended within which the origin lies.

14. As another example of a similar type we will consider the interpretation of the equation

$$\begin{aligned}S.(s_3+s_1)(s_1+s_2) + S.(s_1+s_2)(s_2+s_3) + S.(s_2+s_3)(s_3+s_1)\\ = s_1^2 + s_2^2 + s_3^2 + 3(Ss_2s_3 + Ss_3s_1 + Ss_1s_2).\end{aligned}$$

Let there be three given planes (A), (B), (C), and let (X), (Y), (Z) be the respective polar planes of the origin with respect to (B) and (C), (C) and (A), (A) and (B). Then the above equation becomes

$$\begin{aligned}4S.(Y)(Z) + 4S.(Z)(X) + 4S.(X)(Y)\\ = (A)^2 + (B)^2 + (C)^2 + 3\{S.(B)(C) + S.(C)(A) + S.(A)(B)\}.\end{aligned}$$

Hence if $p_1, p_2, p_3, q_1, q_2, q_3$ be the respective perpendiculars from the origin on (A), (B), (C), (X), (Y), (Z), we have

$$4\left\{\frac{\cos(Y,Z)}{q_2q_3}+\frac{\cos(Z,X)}{q_3q_1}+\frac{\cos(X,Y)}{q_1q_2}\right\}+\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}$$
$$=3\left\{\frac{\cos(B,C)}{p_2p_3}+\frac{\cos(C.A)}{p_3p_1}+\frac{\cos(A,B)}{p_1p_2}\right\}.$$

15. As a final example of formulæ involving products of pairs of planes, we will take the equation

$$S.s_1(s_2'+s_3')+S.s_2(s_3'+s_1')+S.s_3(s_1'+s_2')$$
$$=S.s_1'(s_2+s_3)+S.s_2'(s_3+s_1)+S.s_3'(s_1+s_2).$$

Suppose that we have six given planes (A), (B), (C), (D), (E), (F). Let (L), (M), (N), (X), (Y), (Z) be the respective polar planes of the origin with respect to (B) and (C), (C) and (A), (A) and (B), (D) and (E), (E) and (F), (F) and (D). Then the above equation becomes

$$S.(A)(X)+S.(B)(Y)+S.(C)(Z)$$
$$=S.(D)(L)+S.(E)(M)+S.(F)(N).$$

Hence, if we use p_a to denote the perpendicular from the origin on (A), and adopt a similar notation for the other perpendiculars, we have

$$\frac{\cos(A,X)}{p_ap_x}+\frac{\cos(B,Y)}{p_bp_y}+\frac{\cos(C,Z)}{p_cp_z}$$
$$=\frac{\cos(D,L)}{p_dp_l}+\frac{\cos(E,M)}{p_ep_m}+\frac{\cos(F,N)}{p_fp_n}.$$

16. We will now proceed to the discussion of the scalar portion of the product of three planes, making use of the construction and notation that we adopted in Articles 10 and 11; premising at the same time that PA, PB, PC are to be taken as the edges of that particular solid angle, contained by the three planes, within which the origin lies.

We have the equations

$$S.s_1s_2s_3=S.s_1Vs_2s_3=S.s_2Vs_3s_1=S.s_3Vs_1s_2.$$

Now the perpendicular from the origin on $V.s_2s_3$ is parallel to AP, and $TV(s_2s_3)=\sin A/p_2p_3$. Thus

$$S.s_1Vs_2s_3=-\cos(\pi-\theta)\sin A/p_1p_2p_3,$$

where θ is the angle between PA and the normal to s_1, *i.e.*

$$S.s_1Vs_2s_3=\cos\theta\sin A/p_1p_2p_3=\sin p_a\sin A/p_1p_2p_3.$$

Thus we have

$$S.s_1s_2s_3=\frac{\sin p_a.\sin A}{p_1p_2p_3}=\frac{\sin p_b.\sin B}{p_1p_2p_3}=\frac{\sin p_c.\sin C}{p_1p_2p_3}.$$

With regard to the question of sign, we have

$$S.s_1s_2s_3 = S.s_2s_3s_1 = S.s_3s_1s_2 = -S.s_3s_2s_1 = -S.s_2s_1s_3 = -S.s_1s_3s_2.$$

Draw OL, OM, ON respectively perpendicular to s_1, s_2, s_3. In the above investigation we have proceeded on the assumption that the three planes are so situated that a right-handed screw motion about OP, the direction of the motion of translation being from O towards P, will carry L towards M, M towards N, and N towards L. Thus if the order in which the three planes occur in the product correspond to a right-handed screw motion about OP, then the quantity we have equated to the scalar portion of the product is affected with a positive sign. If, however, the order of the planes correspond to a left-handed screw motion about OP, the said quantity is affected with a negative sign.

Three other expressions for $S.s_1s_2s_3$ may be deduced from the formulae

$$S.s_1s_2s_3 = \frac{1}{Ts_1}TV.Vs_3s_1Vs_1s_2 = \frac{1}{Ts_2}TV.Vs_1s_2Vs_2s_3 = \frac{1}{Ts_3}TV.Vs_2s_3Vs_3s_1.$$

Now the perpendiculars from the origin on $V.s_3s_1$ and $V.s_1s_2$ are respectively parallel to BP and CP, also $TV.s_3s_1 = \sin B/p_3p_1$ and $TV.s_1s_2 = \sin C/p_1p_2$, and consequently we have

$$TV.Vs_3s_1Vs_1s_2 = \sin B \sin C \sin a/p_1^2p_2p_3.$$

Therefore

$$S.s_1s_2s_3 = \frac{\sin B \sin C \sin a}{p_1p_2p_3} = \frac{\sin C \sin A \sin b}{p_1p_2p_3} = \frac{\sin A \sin B \sin c}{p_1p_2p_3};$$

from which we deduce the well-known formula

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C.$$

Another expression for $S.s_1s_2s_3$ is deducible from the formula

$$s_1^2s_2^2s_3^2 + (S.s_1s_2s_3)^2 = s_1^2(Ss_2s_3)^2 + s_2^2(Ss_3s_1)^2 + s_3^2(Ss_1s_2)^2 - 2Ss_2s_3Ss_3s_1Ss_2s_2.$$

From this we deduce

$$(S.s_1s_2s_3)^2 = \frac{1}{p_1^2p_2^2p_3^2}\{1-\cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C\},$$

and combining the three sets of formulæ we obtain

$$\begin{aligned} 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C \\ = \sin^2 p_a \sin^2 A\,^* = \sin^2 p_b \sin^2 B = \sin^2 p_c \sin^2 C \\ = \sin^2 B \sin^2 C \sin^2 a = \sin^2 C \sin^2 A \sin^2 b = \sin^2 A \sin^2 B \sin^2 c.\end{aligned}$$

* See Article 11.

As was the case in Article 10 we see that these formulae may be derived, by aid of the theory of the polar triangle, from the formulae that can be deduced from the same analysis by means of Sir W. Hamilton's method of interpretation*.

17. Suppose that we have three planes s_1, s_2, s_3, and that s is the polar plane of the origin with respect to them. Then $3s = s_1 + s_2 + s_3$, and if we draw a plane s_4 parallel to s, on the opposite side of the origin, and at one-third the distance from the origin, we shall have $s_1 + s_2 + s_3 + s_4 = 0$. Further, since s passes through the point of concurrence of s_1, s_2, s_3, we see that a plane through the origin parallel to s_4 will divide the perpendicular on s_4 from the said point of concurrence in the ratio 3 : 1. From symmetry it is plain that similar statements will be true of the planes through the origin parallel to s_1, s_2 and s_3. Thus it is evident that the origin is the centroid of the tetrahedron $ABCD$ formed by the four planes s_1, s_2, s_3, s_4. Consequently, if p_1, p_2, p_3, p_4 be the perpendiculars from the origin on s_1, s_2, s_3, s_4, we have

$$p_1 \,.\, \Delta BCD = p_2 \,.\, \Delta CDA = p_3 \,.\, \Delta DAB = p_4 \,.\, \Delta ABD.$$

From the equation $s_1 + s_2 + s_3 + s_4 = 0$ we easily deduce that

$$S \,.\, s_2 s_3 s_4 = -S \,.\, s_3 s_4 s_1 = S \,.\, s_4 s_1 s_2 = -S \,.\, s_1 s_2 s_3.$$

Hence, adopting the notation of Article 12, we have

$$\frac{(\Delta BCD)^2}{1 - \cos^2(C, D) - \cos^2(D, B) - \cos^2(B, C) - 2\cos(C, D)\cos(D, B)\cos(B, C)}$$
$$= \frac{(\Delta CDA)^2}{1 - \cos^2(D, A) - \cos^2(A, C) - \cos^2(C, D) - 2\cos(D, A)\cos(A, C)\cos(C, D)}$$
$$= \frac{(\Delta DAB)^2}{1 - \cos^2(A, B) - \cos^2(B, D) - \cos^2(D, A) - 2\cos(A, B)\cos(B, D)\cos(D, A)}$$
$$= \frac{(\Delta ABC)^2}{1 - \cos^2(B, C) - \cos^2(C, A) - \cos^2(A, B) - 2\cos(B, C)\cos(C, A)\cos(A, B)}$$

it being understood that all the dihedral angles contained in these formulæ are interior to the tetrahedron. This result has been given by Wolstenholme.

18. For the sake of convenience we will write the denominators of the four equal ratios of the preceding article as T_1^2, T_2^2, T_3^2, T_4^2, it being understood that T_1, T_2, T_3, T_4 are to be considered positive. Squaring the equation

$$s_1 S s_2 s_3 s_4 - s_2 S s_3 s_4 s_1 + s_3 S s_4 s_1 s_2 - s_4 S s_1 s_2 s_3 = 0,$$

* Tait's *Quaternions*, p. 57.

we have

$$s_1^2(Ss_2s_3s_4)^2+s_2^2(Ss_3s_4s_1)^2+s_3^2(Ss_4s_1s_2)^2+s_4^2(Ss_1s_2s_3)^2$$
$$-2Ss_1s_2\,Ss_2s_3s_4\,Ss_3s_4s_1+2Ss_1s_3\,Ss_2s_3s_4\,Ss_4s_1s_2-2Ss_1s_4\,Ss_2s_3s_4\,Ss_1s_2s_3$$
$$-2Ss_2s_3\,Ss_3s_4s_1\,Ss_4s_1s_2+2Ss_2s_4\,Ss_3s_4s_1\,Ss_1s_2s_3-2Ss_3s_4\,Ss_4s_1s_2\,Ss_1s_2s_3=0.$$

Supposing the origin to lie within the tetrahedron $ABCD$ formed by the four planes s_1, s_2, s_3, s_4, though it no longer necessarily coincides with the centroid, and adopting the notation of the preceding article, we have

$$T_1^2+T_2^2+T_3^2+T_4^2=2T_1T_2\cos(A,B)+2T_1T_3\cos(A,C)$$
$$+2T_1T_4\cos(A,D)+2T_2T_3\cos(B,C)$$
$$+2T_2T_4\cos(B,D)+2T_3T_4\cos(C,D).$$

19. If we square the equations

$$V\,.\,Vs_1s_2Vs_3s_4=s_1Ss_2s_3s_4-s_2Ss_3s_4s_1=s_4Ss_1s_2s_3-s_3Ss_4s_1s_2,$$

we obtain

$$-(TV\,.\,Vs_1s_2Vs_3s_4)^2=s_1^2(Ss_2s_3s_4)^2+s_2^2(Ss_3s_4s_1)^2-2Ss_1s_2\,Ss_2s_3s_4\,Ss_3s_4s_1$$
$$=s_3^2(Ss_4s_1s_2)^2+s_4^2(Ss_1s_2s_3)^2-2Ss_3s_4\,Ss_4s_1s_2\,Ss_1s_2s_3.$$

Now the perpendiculars from the origin on Vs_1s_2 and Vs_3s_4 are respectively parallel to CD and AB*, also $TVs_1s_2=\sin(A,B)/p_1p_2$ and $TVs_3s_4=\sin(C,D)/p_3p_4$. Hence, if (AB, CD) denote the angle between AB and CD*, we have

$$TV\,.\,Vs_1s_2Vs_3s_4=\sin(A,B)\sin(C,D)\sin(AB,CD)/p_1p_2p_3p_4,$$

and our equations become

$$\sin^2(A,B)\sin^2(C,D)\sin^2(AB,CD)=T_1^2+T_2^2-2T_1T_2\cos(A,B)$$
$$=T_3^2+T_4^2-2T_3T_4\cos(C,D).$$

20. Consider next the equation

$$Ss_1s_5\,Ss_2s_3s_4-Ss_2s_5\,Ss_3s_4s_1+Ss_3s_5\,Ss_4s_1s_2-Ss_4s_5\,Ss_1s_2s_3=0.$$

Take the origin within the tetrahedron $ABCD$ formed by the planes s_1, s_2, s_3, s_4, and let the transversal plane s_5 be denoted by the symbol (X). Then we have

$$T_1\cos(A,X)+T_2\cos(B,X)+T_3\cos(C,X)+T_4\cos(D,X)=0,$$

where, as in all former cases, for the dihedral angle between two planes is to be taken that particular dihedral angle within which the origin lies.

* The order of the letters is understood to indicate direction.

21. A construction for the vector portion of the product of three planes can be obtained from the equation

$$V.\,s_1s_2s_3 = s_1Ss_2s_3 - s_2Ss_3s_1 + s_3Ss_1s_2.$$

Draw (X) the mean plane of (A), (B), (C), with respect to the origin, for multiples

$$\cos(B, C)/p_2p_3, \quad -\cos(C, A)/p_3p_1, \quad \cos(A, B)/p_1p_2.$$

Then draw a plane parallel to (X), such that its distance from the origin bears to the distance of (X) from the origin a ratio

$$1 : \frac{\cos(B, C)}{p_2p_3} - \frac{\cos(C, A)}{p_3p_1} + \frac{\cos(A, B)}{p_1p_2}.$$

22. As a final example we will discuss the scalar portion of the product of four planes, supposing, for the sake of fixing ideas, that the origin is within the tetrahedron formed by the planes. We have the equation

$$S.\,s_1s_2s_3s_4 = Ss_2s_3Ss_1s_4 - Ss_3s_1Ss_2s_4 + Ss_1s_2Ss_3s_4,$$

and from this, utilizing the notation of the preceding articles, we deduce

$$S.\,s_1s_2s_3s_4 = \frac{1}{p_1p_2p_3p_4}\left\{\cos(B, C)\cos(A, D) - \cos(C, A)\cos(B, D) + \cos(A, B)\cos(C, D)\right\}.$$

Other expressions may be given for $S.\,s_1s_2s_3s_4$: thus we have

$$S.\,s_1s_2s_3s_4 = Ss_1s_2\,Ss_3s_4 + S.\,Vs_1s_2\,Vs_3s_4 = Ss_2s_3\,Ss_4s_1 + S.\,Vs_2s_3\,Vs_4s_1.$$

Now the perpendiculars from the origin on Vs_1s_2 and Vs_3s_4 are respectively parallel to CD and AB*, also $TVs_1s_2 = \sin(A, B)/p_1p_2$ and $TVs_3s_4 = \sin(C, D)/p_3p_4$. Hence, if (AB, CD) denote the angle between AB and CD*, we have

$$S.\,Vs_1s_2\,Vs_3s_4 = -\sin(A, B)\sin(C, D)\cos(AB, CD)/p_1p_2p_3p_4.$$

Thus we have

$$\begin{aligned} S.\,s_1s_2s_3s_4 &= \frac{1}{p_1p_2p_3p_4}\left\{\cos(A, B)\cos(C, D) - \sin(A, B)\sin(C, D)\cos(AB, CD)\right\} \\ &= \frac{1}{p_1p_2p_3p_4}\left\{\cos(B, C)\cos(D, A) - \sin(B, C)\sin(D, A)\cos(BC, DA)\right\}. \end{aligned}$$

* The order of the letters is understood to indicate direction.

If we equate the different values for $S . s_1 s_2 s_3 s_4$, we obtain

$$\sin(A, B)\sin(C, D)\cos(AB, CD) = \cos(C, A)\cos(B, D) - \cos(B, C)\cos(A, D),$$

and

$$\sin(B, C)\sin(D, A)\cos(BC, DA) = \cos(C, A)\cos(B, D) - \cos(A, B)\cos(C, D).$$

ST JOHN'S COLLEGE,
November 16, 1887.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

THE death of the Rev. C. Trotter, President, occurred after the cessation of the meetings of the Society for the Michaelmas Term. A special meeting of the Council was therefore convened for Tuesday, December 6, 1887, at which the following resolution was passed on behalf of the Society:

Agreed that as a mark of affection for the memory of their late PRESIDENT, the Rev. C. TROTTER, and of deep sorrow for the loss sustained by this Society and the University through his death, a deputation of the Officers and Council of the Cambridge Philosophical Society attend his funeral on Thursday, December 8.

January 30, 1888.

SPECIAL GENERAL MEETING.

PROFESSOR STOKES, VICE-PRESIDENT, IN THE CHAIR.

In consequence of the decease of Mr Coutts Trotter, President of the Society, the following new officers were elected:

President:
Mr J. W. Clark.

Treasurer:
Mr Glazebrook.

Secretary:
Mr Pattison Muir.

Member of Council:
Prof. Babington.

At the Ordinary Meeting which followed, Prof. STOKES, Vice-President, in the Chair, the following communications were made:

(1) *On the Recalescence of Steel, and allied Phenomena.* By H. F. NEWALL, M.A., Trinity College.

IT is very striking how great are the differences in physical properties produced by minute differences in composition of specimens of iron and steel. The magnetic properties of iron are of such importance that it hardly needs pointing out that phenomena occurring nearly simultaneously with the birth of marked magnetic susceptibility are well worth studying.

A piece of steel heated up to a white heat and allowed to cool shews, at certain points in the cooling, peculiarities recognized as occurring within a small range of temperature about red heat.

It becomes brighter or reglows, instead of continuing to get gradually darker (Barrett, *Phil. Mag.*, Vol. 46, 1873).

It expands instead of continuing to contract (Gore, *Proc. R. S.* XVII., 1869).

It becomes magnetic (Gilbert, Fox, Faraday and others).

It shews changes in its thermo-electric relations (Tait, *Trans. R. S. Edin.* XXVII., 1873).

Its electrical resistance changes (Smith, Knott and Macfarlane, *Proc. R. S. Edin.*).

It ceases to be capable of assuming on sudden chilling the condition known as glass hardness (Chernoff, Strouhal and Barus).

Its specific heat changes abruptly (Pionchon).

Its rigidity shews marked peculiarities (Tomlinson, Newall, *Phil. Mag.* XXIV., 1887).

Its viscosity shews marked changes.

Many of these changes point to a change in the steel as marked as a chemical change would be. Some of the peculiarities are such as may be explained by a rise of temperature in the cooling steel.

Recalescence or Reglow.

The most striking way of shewing the darkening and recalescence of steel is by heating a thin steel plate with a large blow-pipe flame. As heating proceeds, the plate becomes red hot: then a dark patch appears in the hottest part: this extends and opens into a dark ring, which expands, enclosing a constantly brightening patch. If the heating is now stopped, and cooling allowed to take place, the luminosity of the whole hot part diminishes: there appears on the confines of the part lately occupied by the dark ring a bright ring, which contracts and becomes a

bright patch in the middle of the hottest part of the plate and finally disappears altogether. These appearances I have observed in plates $\frac{1}{8}$ in. thick, but most markedly in the very thin plates used in the manufacture of pens. Messrs Gillott of Birmingham were kind enough to present me with several of these thin sheets.

Barrett, who first records observations on recalescence, failed to observe it in wires of small diameter or in bars of large diameter. I have observed it in steel plate ·1 mm. thick, and in steel spindles over 13 mm. in diameter. I have no reason to believe that the phenomenon does not occur in the largest ingots of steel.

Barrett does not seem to have satisfied himself completely of the existence of a corresponding effect in the heating. I have described above the appearance of the dark patch and expanding ring, and have no difficulty in observing this 'darkening' even in wires ·5 mm. diameter, though it is most markedly seen in the sheet steel.

Prof. Barrett shewed by enclosing a wire in a kind of air thermometer that at the moment of recalescence in cooling there was an increase of pressure in the air surrounding the wire. I have shewn by putting a thermo-electric couple into a small hole in a thick bar of steel, that there is a rise of temperature throughout the mass of the steel.

Prof. G. Forbes (*Proc. R. S. Edin.* VIII. 363, 1876) has attempted an explanation of the reglow, roughly as follows. Iron is a bad conductor of heat—worse above than below dull red heat (Tait). Radiation from the surface carries off heat more rapidly at first than it is supplied from within by conduction. As cooling proceeds, the radiation diminishes, conductivity increases. At a certain point in the cooling the supply from within is greater than is disposed of by the surface by radiation. Hence the surface rises in temperature and will even reglow. Prof. Forbes also gave reasons why the effect was not observed in very small or in very large pieces of metal. There must be some thickness that the difference of temperature within and without may be great enough: this is not attained in thin wires; there must be surface enough relatively to mass, and convection currents enough, to bring about the cooling of the radiating surface: this is not attained in large masses of metal.

I have observed recalescence and darkening in steel whose thickness is less than ·1 mm. Hence we must believe that there can be very considerable difference of temperature in ·05 mm. of iron, or else give up the ingenious theory suggested by Forbes.

I find that some soft iron does not shew reglow at all: that in some hard iron, it is very difficult to observe it because of its

very short duration: but in most steel, especially hard steel, it is peculiarly marked.

To test whether chemical action at the surface has anything to do with the phenomenon, I have heated wires of hard steel in nitrogen and in vacuo and have still observed recalescence undiminished. Hence we must conclude that recalescence is not to be attributed either to oxidation or to occlusion of gases.

Expansion of steel during recalescence.

Dr Gore stretched iron wires in a special expansion apparatus and observed that at a certain point in cooling from a bright red heat the wire instead of continuing to contract regularly, suddenly expanded, and then contracted continuously. He thought that the wire to shew this expansion must be under tension and attributed the phenomenon to alteration in cohesion.

Prof. Barrett shewed that this tension was not an essential part and that it even prevented part of the phenomenon, namely a contraction during heating, from being seen.

In my own experiments I have found it convenient to have the wires under a slight pressure, but have necessarily had to be careful that the pressure did not bring about curvature, a very small amount of which would easily absorb, so to speak, the expansion looked for. I have confirmed Prof. Barrett's observation of expansion simultaneous with recalescence and have further observed a contraction during heating occurring simultaneously with darkening or diminution of luminosity in spite of constant application of heat.

The expansion during recalescence, one is compelled to believe, must be very considerable: for the reglowing part may be made very small compared with the rest of the length of the rod without doing away with the balance in favour of expansion.

If a steel rod is put into a brass tube which completely encloses it, and the brass tube is put into an expansion apparatus and heated to a bright red heat, the contraction of the brass tube in cooling is attended by the same peculiarities as that of the steel rod, shewing the accession of heat very markedly at a certain point of the cooling.

Glass hardness of steel produced by sudden chilling.

It is known that glass hardness can only be produced in steel if the temperature from which the steel is chilled is higher than a certain temperature about dull red heat.

I observe that a bar of steel cannot be made glass hard unless its temperature is raised above the temperature at which 'darkening' takes place. This is I believe a new observation.

It is a general practice in hardening steel to raise it to a bright red heat and chill immediately. I find that so long as it has been heated above 'darkening', it need not be chilled until just before reglow takes place. Chilling during reglow brings about a considerable degree of hardness: but chilling after the reglow leaves the steel as tough as though it had not been heated above the critical temperature at all. It may be that this temperature is Chernoff's temperature '*b*'.

Steel in which reglow is very marked very often cracks in hardening, especially if not heated very high before quenching. Such steel would be very interesting from the point of view of reglow: but the manufacturers clearly have no interest in making such steel.

Chilled steel is, I think, recognised to have smaller density than the same steel before chilling. This is to be referred to the cooling of the outer rind on a swelled core, which cools and contracts later but leaves the external dimensions larger than they would be had the cooling been a slower process. In the case of bad steel, cracking by hardening is probably to be referred to actual expansion of the reglowing core within a cooled rind. The importance of this peculiarity from a practical point of view in the treatment of steel may well be considerable.

Thermo-electric peculiarities of steel.

A study of the line for steel in Prof. Tait's diagram shews that the peculiarities about a dull red heat could be partly explained if we could shew that the temperature of the steel fell at a certain point in the heating in spite of continued application of heat, and after falling for some time, then rose again continuously, whilst the thermometer in the enclosure containing the steel couple shewed continuous rise of temperature. We should, if this were the case, have to cut out a vertical strip so far as iron is concerned and close up the interval.

If the peculiarity in steel is partly attributable to irregularities in heating or cooling, then a thermo-electric couple of, let us say, Pt Cu heated in a steel tube will shew the peculiarities ascribed by Prof. Tait to a change of sign in the Thomson specific heat of electricity. I have made this experiment and it shews most decisively that the peculiarity is not thermo-electric, or at least not wholly so.

A piece of steel wire about 1 cm. diameter and 7 cms. long was softened and a hole 1 mm. diameter and 3 cms. long was drilled down its axis. A thermo-electric couple of Pt Cu composed of wires No. 36 B. W. G. was inserted, precautions being taken that there should be contact with the tube only at the

junction of the Pt Cu couple, if at all. The free ends were attached to a galvanometer which was most satisfactorily dead beat, a vane dipping into water being attached to the magnet and mirror. The galvanometer indications during heating denoted a continual rise of E.M.F. in the circuit until the steel became red hot; then there came a slight *fall* of E.M.F. coincident with 'darkening' in the steel; and then again a continuous rise until the temperature of the steel became steady. In cooling, the fall of E.M.F. in the circuit went on continuously until after diminishing in rapidity the fall changed into a *rise* coincident with the reglow in the steel tube. In making this experiment I met with considerable difficulties before success; my difficulties were due to my having at first taken too small a steel tube in which to enclose the Pt Cu couple, so that the thick wires of the couple conducted away the heat too quickly for any actual rise in the reglow to be observable—the pause was obvious enough. Care was taken to shew that the galvanometer indications were not due to insufficient damping.

We must regard this experiment as shewing that the steel tube though put into conditions that promote continuous heating or continuous cooling does not either rise or fall in temperature continuously.

The galvanometer indications denote that there is a fall of temperature coincident with darkening during heating, and a rise of temperature coincident with reglow during cooling, and that the temperature of darkening is higher than that of reglow. In the latter point care has been taken, by reducing the rate of heating and cooling, to shew that the indications were not to be attributed to inertia in the thermometric arrangements but are really signs of something of the nature of what Prof. Ewing calls hysteresis.

If a piece of steel wire is connected up with a galvanometer and these connections kept at constant temperature, then if at any intermediate part of the wire a part be heated there will be no resultant electromotive force unless there is want of homogeneousness or of isotropy or of similarity of magnetic state. Let the heating be continued so that the part is heated above the temperature of darkening; still no E.M.F. results, if the heated part is kept the same. If however this part is shifted along the wire so that darkening takes place in front and reglow behind an E.M.F. results, of the order of a milli-volt, and persists only so long as shifting continues. The fact that the temperatures of darkening and reglow are different shews that we must consider the steel between the points at which reglow and darkening are taking place as different from that outside those points. It will be convenient to speak of this as altered steel or 'hot' steel as

opposed to 'cold' steel. This E.M.F. cannot be explained by reference to the Thomson forces, unless they are of such a kind that summed round a circuit they do not disappear.

Magnetic susceptibility in cooling steel and iron.

Many experiments have been made on this point: but it appears that in far the greater number of these no pure results could be obtained, because the magnetometer used has in general been too refined for the method of heating and cooling employed.

There can be little doubt that above a certain temperature steel and iron do not differ from air in magnetic susceptibility. The doubt is as to the mode of appearance as temperature falls. Most experimenters find that there is at first a gradual accession, then a sudden rush, so to speak, and afterwards a gradual increase of magnetic susceptibility. My own impression is that the first gradual accession is in reality due to unequal cooling. The greater the precautions to secure equal cooling throughout the mass of steel used, the more does the first gradual accession diminish.

I have experimented with small spheres of metal sunk into a thick brass rod, the rod being cut in two, the ends faced and a hemispherical hole made in each, so that the sphere of steel or iron might be gripped between the two faces and heated in the brass.

The appearance, during cooling, of magnetic susceptibility is very much more leisurely in steel than in soft iron, and it is most curious to observe how the *rate* of appearance entirely corresponds with the rate of reglow. In all cases where I failed to detect reglow, the iron became magnetic with a flash, as it were, so that the magnetometer needle was deflected with a kick, carrying it much beyond the new position of rest. With specimens of hard steel in which reglow was very slow, the magnetometer needle moved slowly and almost without any swinging up to its new position.

A strong steel bar magnet was held perpendicular to the magnetic meridian and the small magnetometer needle was then set so as to be almost at the neutral point in the resultant field of the bar magnet and Earth's horizontal component, only enough displacement being given to direct the needle conveniently for lamp and scale reading. Then an unmagnetised sphere of soft iron was moved about near the magnetometer, and the spot was settled experimentally where the maximum effect on the magnetometer needle was observed. This spot was marked by lining with fixed objects, and the brass rods fixed

so that the experimental sphere occupied that spot. Heating was done by means of a Bunsen burner flame.

Beyond the observation of correspondence of the character of reglow with that of the appearance of magnetic susceptibility, my results are not yet of such a kind as to warrant publication.

General remarks.

Many experimenters have shewn that there are as marked differences between steel above and below a certain temperature about red heat as there are between steel and other metals recognised as different.

We may represent the different properties and their variations with temperature graphically, and we evidently deal with one and the same substance in steel with rise of temperature until darkening occurs. Then in the graphical representation we pass from one line to another and for certain limits of temperature about red heat there are two values of the ordinate for each value of the temperature abscissa, the limits being those corresponding to the temperatures of reglow and darkening. We can trace the curve for cold steel in heating up to darkening, and that for hot steel in cooling down to reglow.

There is an actual *diminution* of luminosity, an actual *fall* of temperature, not only a pause in the increase, as heat is applied to steel from without. As steel is heated, a certain point is reached, when some action—it may be physical or it may be chemical—begins and continues with the "violence of instability". Either it is a case similar to that of water heated above its boiling point and suddenly bursting into vapour with a fall of temperature down to the boiling point: or it is some chemical decomposition of unusual type, which should have begun at a lower temperature, but which having been delayed continues, when once started, like an explosion. I cannot find reference to any such chemical action. Prof. Liveing has pointed out several chemical actions accompanied apparently with a fall of temperature, but has explained these by shewing that the fall is due to physical, not chemical changes, volatile compounds being formed in certain cases at temperatures above the boiling points, so that when formed they boil away with reduction of temperature.

The only apparent chances for explanation by reference to chemical action are not hopeful:—carbides of iron, and allotropic changes—I hesitate to name ferricum and ferrosum.

Reglow in cooling is somewhat similar in aspect to a phenomenon, to which Prof. Dewar kindly called my attention. In gold assaying it is often observed that a molten button of gold may be cooled below its solidification point without solidifying.

When solidification does take place, there is a rise in temperature and as the point of solidification is above red heat, the button reglows or flashes, as it is called. The explanation is obvious, though it is not clear why a trace of lead should make such marked differences.

The chances for explanation of the recalescence of steel by reference to surfusion are as little promising as those by reference to chemical action.

(2) *On the Method of Measuring Surface Tension by observations of the form of liquid drops.* By H. L. CALLENDAR, B.A., Trinity College.

(3) *On the arrangement of electrified cylinders when attracted by an electrified sphere.* By JAMES MONCKMAN, (D.Sc. London), Downing College.

At the suggestion of Professor Thomson I have endeavoured to solve experimentally the question of distribution of a number of electrified cylinders parallel to each other when acted upon by an electrified sphere placed above them.

The simplest thing would be to place a metallic cylinder such as a wire, or knitting needle, through a flat piece of cork or wood, and leaving one portion sufficiently long to cause the other to rise vertically out of the water. This plan however does not succeed. Firstly because there is sufficient attraction between water and cork to draw the floats into sets of two or more and to hold them together, when once they approach nearer than a certain point; secondly they move too slowly through the water; thirdly, whenever any considerable portion of the float is above the water, the electricity upon it causes serious disturbances, and when the portion giving buoyancy is a central mass with a cylinder of high specific gravity passing through it, we find that it is almost impossible to regulate the weight so as to sink the float without sending the whole to the bottom of the water.

Finding that this was so, I decided that the portion below the water should be of metal, the portion out of the water as light as possible, and the float globular and of glass. After several attempts an article has been made (fig. *a*) which acts fairly well. It consists of a shawl-pin about two inches long pushed through a small disc of lead, then through a thin globular glass bead of about half an inch diameter. This is made water-tight by means of wax. The pin is then pushed through a second smaller bead and waxed, the point projecting about a quarter of an inch. Upon this is fixed the stick of a match previously

dipped into nitrate of silver and exposed to light and the fumes of phosphorus to render it a good conductor of electricity.

To prevent a flow of electricity from the matches to the ball the ends are covered with wax. It is essential that all the matches rise to the same height above the surface of the water or the larger ones force their way to the centre and spoil the figures. To achieve this the weight of the pin and disc must be very carefully adjusted by cutting the lead if too heavy and by adding white lead if too light.

Two large Leyden jars were introduced into the circuit to prevent the charge falling too quickly when the print was being taken.

I found that the best method of printing was to cover one end of a long strip of glass with dilute printer's ink and touch the top of each match with it. When the whole had assumed the proper figure, a sheet of glass was slipped in between the ball and the floats and gently lowered until good contact was made, after which it was transferred to paper.

The figures obtained are shewn in the diagram and are placed so that the line of symmetry is vertical. (Figures to which M is put were produced by an electromagnet only.)

The next experiment was to repeat A. M. Mayer's experiment with magnets floating on water (*American Journal of Science and Art*, 1878, 3 Series, V. XVI., page 247), in order to compare the two sets of figures. The figures obtained varied somewhat from his. This is caused by the change of distance of the attracting magnet. Thus with eight floating magnets a very small attracting magnet gave three in the centre and five outside, a larger one gave figure 8 *b*, and a larger still further removed again fig. 8 *a*. Others vary in the same way.

In order to get a more even field Prof. Thomson proposed to use a large electromagnet. This being placed at a distance of about two feet caused several changes in the combinations.

The figures were more regular than those given by the electrified floats. I did not however consider it necessary to print them when they produced the same forms.

4 to 9 magnets gave 4 to 9 (in fig. 2), 8 gave 8*a*, 9 and 10, 9, 10*a*, 10*b*, 11 gave 11*a*, 12, 12*a* and M 12, 13 and 14 gave 13 and 14 and 15, 15*b*, 18 and 20 gave 18 and 20, but 16, 17, and 19 gave M 16, M 17 and M 19.

In some of the higher numbers a tendency to set themselves in hexagons with one in the centre was observed, and as this is the figure in which the force on each magnet is equal, it appeared probable that with an even magnetic field and an infinite number of floating magnets the whole would be thus distributed. To test this forty-eight needles were prepared and placed upon the

water, and when the current was passed the central floats took that form, which is exactly that of M 19. In cases such as 9 where there are not sufficient magnets to form two hexagons, pentagons, the nearest form, are produced either alone or mixed. Or reducing the figures to triangles the perfect form is an equilateral triangle which is produced when the numbers are great enough and when the pressure is equal on all sides; failing numbers, the triangles of the pentagon with angles of 72°, 54° and 50° are given. Pressure not being equal, the triangles become irregular.

I give figures 9, 10 and M 19 so marked as examples. The figures produced by the electrified cylinders are not as regular as they might be got with better floats, but it is evident that they vary from those given by magnets, only, on account of their imperfection, and that they obey the same law and with perfect floats would produce identical shapes.

February 13, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

W. BATESON, M.A., Fellow of St John's College, was elected a Fellow of the Society.

The following communications were made:

(1) *On Variations of Cardium edule from the Aral Sea.* By W. BATESON, M.A., St John's College.

The author stated that he had visited the Aral Steppe in order to see if any structural variations could be found in animals living in water of different degrees of saltness which could be correlated to those conditions. In the case of the cockle an instance of this had occurred. It is well known that the level of the Aral Sea has fallen in recent times, and it is certain that it formerly stood about 15 feet higher than it now does. The water then covered a considerable belt of low lying country to the North and East of the present shore line. This region is shewn by the presence of vast numbers of shells of the cockle, which still lives in the Aral Sea. In the area which is now exposed lie three depressions, Jaksi Klich, Jaman Klich and Shumish Kul, which are now dry lakes with a crust of salt at the bottom. When the Aral Sea retreated these basins must have remained for some time as isolated salt lakes in which the cockles were confined.

In the case of Shumish Kul the sides of the lake have been terraced probably by the oscillation of the water under the wind. On these terraces the cockles are found, generally as paired valves with their ligaments more or less preserved, sitting up in their natural attitude. As the lake dried up, the water must have become salter; hence the cockles on any one terrace must have lived in salter water than that which was inhabited by the cockles of the terrace above it. Passing from above downwards a progressive series is thus obtained. On comparing samples from each terrace it appears that under the influence of increasing salinity the shells (1) became thinner (ten shells from the lowest bed weigh 1·54 gr. while ten shells of the same size from the highest weigh 5·18 gr.), (2) their length in an antero-posterior direction increased relatively to their dorso-ventral width, (3) they became much more highly coloured, (4) only the lowest shells shew diminution in absolute size, (5) deformed shells are very common on the lower terraces.

The chief interest of these facts lies in the almost perfect uniformity of each sample as regards texture and colour. The uniformity in shape is not so great. Even amongst the longest shells individual rounder ones occur, but if measurements of a number be taken together, the increase in average length is considerable.

The same changes are visible in the case of the cockles in Jaksi Klich and in Jaman Klich, though these lake-beds shew hardly any terracing. As all these three lakes were isolated from each other, it may be concluded that the variations observed occurred as the specific consequence of the changed conditions.

(2) *The character of the geological formation a factor in Zoogeographical Distribution, illustrated by observations in Portugal and Spain.* By H. Gadow, M.A., King's College.

The observations concern the distribution of all the species of Amphibia and Reptiles in the Peninsula, based upon a great number of data.

The method adopted is the following. The various species and localities are arranged in six classes, which represent the most typical geological formations, so far as the purpose of the investigation is concerned. The proportionate areas of the various formations are given in per cent. of the whole country.

On Granitic ground, equal to 29 per cent. of the total area of Portugal, 65 localities for Amphibia were recorded. On 42 per cent. there should be 94, but there are only 15 on record, these 42 per cent. being Palaeozoic. Consequently Granitic ground is about six times more favourable to Amphibian life than is Palaeo-

zoic. In this way it was found that the various formations range as follows, concerning Amphibia: New Red Sandstone and Granitic are by far the most favourable, then follows Tertiary and Palaeozoic terrain, and lastly as very unfavourable Jurassic and Cretaceous.

Very different results are obtained for Reptiles. New Red Sandstone is likewise the best, but then follows Jurassic, Granitic, Tertiary and lastly Palaeozoic.

In both cases Palaeozoic ground is unfavourable. Certain species are almost absolutely restricted to certain formations, whilst their distribution is independent of Isothermes or altitude.

The reptilian life in Jurassic districts is restricted to the insectivorous Lizards and to Lizard-eating snakes. The probable interdependence between the Geological formations, the resulting amount of water, and the Flora and Fauna, was touched.

Under certain circumstances a small strip of rather low Palaeozoic or Jurassic ground may form a much more effective barrier than very high and extensive granitic mountain ranges.

(3) *Note on the Physiology of Sponges.* By G. P. Bidder, B.A., Trinity College.

After feeding with suspended carmine a calcareous sponge (*Leucandra aspera,* Vosmaer) the author found that in it the carmine was at no time in any but the collared cells. The water is filtered of the particles suspended in it by a membrane, formed by the coalescence of the collars, which stretches completely across the current. This coalescence has been figured by Sollas in certain siliceous sponges. The whole evolution of the canal-system in sponges consists in increasing the energy of the oscular flow and diminishing the velocity in the flagellate chambers. In these are alike specialized the functions of absorption and propulsion, since to each a low velocity is advantageous. The author believes that the collared cells primitively both ingest and digest the food, the collars having as their function its retention; digestion is only secondarily passed to the mesoderm.

(4) *On the development of Aleurone grains in the Lupine.* By A. B. Rendle, B.A., St John's College.

The development of Aleurone grains was studied by Pfeffer, sixteen years ago, and according to his results is as follows. The mineral constituents, 'globoids' of lime and magnesia phosphate, or crystals of calcium oxalate, appear in the cell sap, and act as centres of attraction for the proteid matter, which is also precipitated from the turbid cell sap, and by aggregating round the inorganic matter forms the aleurone grain.

It would appear however from the study of development of these structures in species of Lupine, that the process is by no

means so mechanical but consists in an active secretion by and in the protoplasm. Sections of seeds which are just beginning to swell are seen to contain a number of more or less spherical bodies protruding from the protoplasm and staining deeply with iodine, Hofmann's blue and other dyes which stain protoplasm, and by treatment with dilute KOH, these are shewn to be due to the secretion of some substance, presumably proteid, soluble in dilute KOH and separated by a delicate protoplasmic membrane from the vacuole. The substance is insoluble in solutions of NaCl or dilute HCl. By increase in size the little masses of secretion soon fill up the whole interior of the cell, but always remain separately embedded in a protoplasmic reticulum.

The substance originally secreted appears to break down shortly into the several proteids found in the ripe grain, a process which is indicated by alteration in solubility, as by the time the vacuole has been filled up the grains have become soluble in salt solutions and in fact behave with regard to solvents exactly as those of the ripe seed. It is probably during this chemical decomposition that the mineral matter appears, but in the species which was specially studied, *L. digitatus*, the grains even in the fully ripe seed, contain no solid mineral matter at all.

It is interesting to note that the process of development indicated here very closely agrees with that lately described by Gardiner and Ito for mucilage drops in the secreting hairs of Blechnum and Osmunda.

February 27, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

H. L. Callendar, B.A., Fellow of Trinity College.
T. C. Fitzpatrick, B.A., Fellow of Christ's College.
G. P. Bidder, B.A., Trinity College.

The following communications were made:

(1) *On a Radiation Problem.* By E. W. HOBSON, M.A., Christ's College.

If a uniform rod of section so small, that the motion of heat in it may be regarded as linear, be bounded by the plane $x=0$ and extend to an indefinite distance in the positive direction of x, the formula

$$v=\frac{x}{2\sqrt{\pi\kappa}}\int_0^t \frac{f(\lambda)}{(t-\lambda)^{\frac{3}{2}}}e^{-\frac{x^2}{4\kappa(t-\lambda)}}d\lambda \dots\dots\dots\dots(1),$$

in which κ is the conductivity of the rod divided by its density and specific heat, represents the temperature at any point of the rod when the initial temperature is zero and the end $x=0$ is maintained at temperature $f(t)$; writing $\frac{x^2}{4\kappa(t-\lambda)}=q^2$ this formula becomes

$$v=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2q\sqrt{\kappa}}}^{\infty} e^{-q^2}.f\left(t-\frac{x^2}{4\kappa q^2}\right)dq \text{ ....(2).}$$

Sir W. Thomson has shewn ('On the Electric Telegraph,' *Collected Works*, Vol. 2, No. LXXIII.) that these formulae may be obtained by supposing a doublet consisting of a source and sink to be placed at the point $x=0$, the rod being supposed to be continued on the negative side of the origin; the product of the amount of heat generated in time dt by the source, into the distance between the source and sink, is equal to $2\kappa f(t)\,dt$; the formulae (1) and (2) represent the temperature due to this doublet.

In the present communication formulae are given corresponding to (1) and (2) for the case in which instead of the temperature at the plane $x=0$ being given, radiation takes place across that plane into a medium of which the temperature is given, say $f(t)$. In this case the boundary condition which holds when $x=0$ is

$$\frac{dv}{dx}=h\{v-f(t)\},$$

where h denotes the external conductivity. Let $u=v-\frac{1}{h}\frac{dv}{dx}$, then

$$\left(\frac{d}{dt}-\kappa\frac{d^2}{dx^2}\right)u=\left(1-\frac{1}{h}\frac{d}{dx}\right)\left(\frac{d}{dt}-\kappa\frac{d}{dx^2}\right)v=0;$$

thus u satisfies the equation of conduction and can be determined from the conditions that $u=0$ when $t=0$ and $u=f(t)$ when $x=0$; in fact u is given by either of the expressions (1) and (2).

We have $$\frac{d}{dx}(ve^{-hx})=-hue^{-hx},$$

hence $$v=he^{hx}\int_x^{\infty}ue^{-h\xi}d\xi,$$

the upper limit is infinite because v must vanish when x is infinite; let $\xi=x+z$, then the expression for v becomes

$$\int_0^{\infty}he^{-hz}u_1dr \text{ ....(3),}$$

where u_1 denotes what u becomes when $x+z$ is written for x; the required expressions for v are then

$$v=\frac{h}{2\sqrt{\pi\kappa}}\int_0^\infty\int_0^t\frac{(x+z)f(\lambda)}{(t-\lambda)^{\frac{3}{2}}}e^{-hz}.e^{-\frac{(x+z)^2}{4\kappa(t-\lambda)}}dz\,d\lambda\ldots\ldots(4)$$

$$=\frac{2h}{\sqrt{\pi}}\int_0^\infty\int_{\frac{x+z}{2q\sqrt{\kappa}}}^\infty e^{-q^2-hz}f\left\{t-\frac{(x+z)^2}{4\kappa q^2}\right\}dz\,dq\ \ldots\ldots\ldots(5).$$

As in the case above mentioned in which the temperature at $x=0$ is given, the rod may be replaced by a rod, infinite in both directions, with a doublet at the origin, so in the present case the rod may be supposed to be continued on the negative side of the origin with a distribution of doublets along the whole of the part of the rod so continued; in fact the equation (4) shews that if, in the infinite rod, doublets be placed of magnitude $2h\kappa e^{-hz}.dz.f(t)$ at the element dz which is at the point $x=-z$, the temperature at any point on the positive side of the origin due to all this continuous distribution of doublets will be the actual temperature of the given semi-infinite rod under the given radiation conditions supposing the initial temperature of the rod to be zero. The temperature due to the doublet at $x=-z$ is

$$\int_0^t\frac{h}{2\sqrt{\pi\kappa}}dz\frac{(x+z)}{(t-\lambda)^{\frac{3}{2}}}.e^{-hz}.e^{-\frac{(x+z)^2}{4\kappa(t-\lambda)}}f(\lambda)\,d\lambda,$$

and the complete temperature is that obtained by summing this expression for all values of z from 0 to ∞. Hence corresponding to Thomson's single doublet at the origin, there must be in the radiation problem a distribution of doublets on the whole of the negative part of the rod, and their magnitude diminishes in geometrical progression with the distance from the origin.

Hitherto it has been supposed that the initial temperature over the semi-infinite rod was zero, but the formula (3) may be applied to obtain the expression which must be added to the expression (4) or (5) when the initial temperature is arbitrarily given, say, $\phi(x)$. In this case u must be determined from the conditions $u=\left(1-\frac{1}{h}\frac{d}{dx}\right)\phi(x)$, when $t=0$ for all positive values of x, and $u=0$ when $t=0$; now the expression

$$\frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty\left\{e^{-\frac{(x-x')^2}{4\kappa t}}-e^{-\frac{(x+x')^2}{4\kappa t}}\right\}\psi(x')dx'$$

is the temperature due to a series of sources of magnitude $\psi(x)\,dx$ on the positive side of the origin and of sinks of magnitude

$-\psi(x)\,dx$ on the negative side of the origin; it therefore satisfies the equation of conduction, vanishes when $x=0$, and is equal to $\psi(x)$ when $t=0$ and x is positive; putting $\psi(x)=\phi(x)-\frac{1}{h}\phi'(x)$, we have therefore the value of u, hence the required value of v is from (3),

$$\frac{h}{2\sqrt{\pi\kappa t}}\int_0^\infty\int_0^\infty e^{-hz}\left\{e^{-\frac{(x+z-x')^2}{4\kappa t}}-e^{-\frac{(x+z+x')^2}{4\kappa t}}\right\}\left\{\phi(x')-\frac{1}{h}\phi'(x')\right\}dz\,dx' \quad (6).$$

The sum of the expressions (4) and (6) is the complete value of v which is initially equal to $\phi(x)$ when x is positive, and which satisfies the condition $\frac{dv}{dx}=h\,[v-f(t)]$ when $x=0$.

The expression (6) shews that the effect of the initial temperature $\phi(x')$ over an element dx' may be thus represented; suppose the rod extended to infinity on the negative side of the origin and suppose an infinite series of instantaneous sources stretching from the point $x=x'$ to $x=-\infty$; the strength of the one between the points z and $z+dz$ being $he^{-hz}dz\left\{\phi(x')-\frac{1}{h}\phi'(x')\right\}dx'$, z being the distance of a source from the point $x=x'$; and also suppose a series of sinks stretching from $x=-x'$ to $x=-\infty$ of the same strength at the distance z from this latter point as the source at the same distance from $x=x'$; such a distribution of sources and sinks will produce a temperature which satisfies the given condition at the point $x=0$ and has the given initial value.

It has thus been shewn that the complete solution of the radiation problem, which is given by the sum of the expressions in (5) and (6), is a solution representing the temperature in a rod indefinitely extended in both directions due to a system of doublets, sources and sinks, the magnitudes of which are completely determined.

The Fourier solutions may of course be deduced from the definite integrals in (4) and (6).

(2) *Notes on Conjugate Functions and Equipotential Curves.* By J. Brill, M.A., St John's College.

1. Suppose that we have a certain function of a complex variable, $w=f(z)=\xi+i\eta$, then it follows that

$$\frac{dw}{dz}=f'(z)=\frac{\partial\xi}{\partial x}-i\frac{\partial\xi}{\partial y};$$

and, therefore,

$$\log \frac{dw}{dz} = \log \left\{\frac{\partial \xi}{\partial x} - i \frac{\partial \xi}{\partial y}\right\}$$

$$= \frac{1}{2} \log \left\{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right\} - i \tan^{-1} \left\{\frac{\partial \xi}{\partial y} \Big/ \frac{\partial \xi}{\partial x}\right\}$$

$$= \log h - i\vartheta.$$

If we interpret ξ as the velocity potential and η as the stream function of a case of steady, irrotational, two-dimensional, fluid motion, then h is the velocity of the fluid at any point, and ϑ is the angle which the direction of motion at that point makes with the positive direction of the axis of x.

Let w_1 and w_2 be two functions of a complex variable, and let

$$(m+n) \log \frac{dW}{dz} = m \log \frac{dw_1}{dz} + n \log \frac{dw_2}{dz},$$

or

$$\frac{dW}{dz} = \left(\frac{dw_1}{dz}\right)^{\frac{m}{m+n}} \left(\frac{dw_2}{dz}\right)^{\frac{n}{m+n}}.$$

Then, obviously, W will be a function of a complex variable, and may be considered as giving rise to a possible case of fluid motion. Let ϑ be the angle made by the direction of motion at any point of the fluid in this case with the positive direction of the axis of x; also, let ϑ_1 and ϑ_2 denote the corresponding quantities for the cases of fluid motion that may be derived from w_1 and w_2; then we have

$$(m+n)\,\vartheta = m\vartheta_1 + n\vartheta_2.$$

Now suppose that we have the stream lines of these latter two cases of motion traced upon a plane, then ϑ_1 and ϑ_2 will be the angles made with the positive direction of the axis of x by the tangents to two stream lines, one belonging to each system, at their point of intersection. Also, ϑ is the angle made with the same line by the tangent at the said point to the particular stream line of the derived case of motion that passes through that point. Obviously, this last tangent divides the angle between the other two in the ratio $n : m$. Thus we have the following theorem:

Two families of equipotential curves are traced on a plane. A third family of curves is drawn possessing the property that the tangent at any point to any member of the family divides in a constant ratio the angle between the tangents at that point to the particular members of the two former families that intersect in the said point. This last family of curves will also form an equipotential system.

If for the two first-mentioned families we take the two conjugate families derived from the same function of a complex variable, this reduces to the well-known theorem:

The isogonal trajectories of a family of equipotential curves themselves form an equipotential system.

It is evident that our theorem may easily be extended to suit the case where we have n families of equipotential curves traced on the plane. In this case we should have n curves intersecting in a given point; and $\vartheta_1, \vartheta_2, \ldots\ldots, \vartheta_n$ would be the angles made by the tangents to these curves, at their point of intersection, with the positive direction of the axis of x. The angle made with this same line by the tangent at the given point to the particular curve of the derived system that would pass through it, would be

$$\frac{m_1\vartheta_1 + m_2\vartheta_2 + \ldots\ldots + m_n\vartheta_n}{m_1 + m_2 + \ldots\ldots + m_n},$$

where $m_1, m_2, \ldots\ldots, m_n$ are given constants. And, if we wish to generalize still further, we may make this last angle equal to

$$\frac{1}{p}(m_1\vartheta_1 + m_2\vartheta_2 + \ldots\ldots + m_n\vartheta_n),$$

where p is another given constant.

2. With the aid of the theorems of the preceding article, we may obtain several of the known systems of equipotential curves by combining systems of concentric circles. Thus, suppose that we have two systems of concentric circles, S and S' being the centres of the two systems. Then, if two circles, one belonging to each system, intersect in the point P, the bisectors of the angles between the tangents at P to the two circles will bisect externally and internally the angle between SP and $S'P$. Thus we should have two families of confocal conics; and, since these two families are orthogonal, it is evident that they are both derivable from the same function of a complex variable. Now a series of concentric circles, having their centre at the point $(a, 0)$, is given by the equation $w = \log(z-a)$, from which we deduce

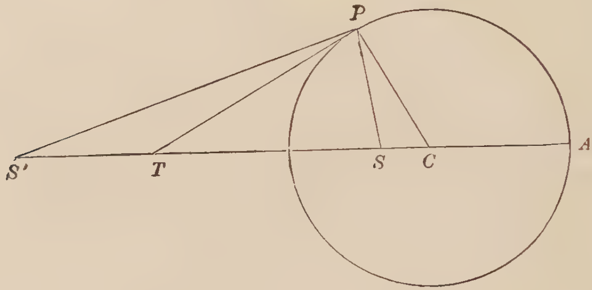
$$\log\frac{dw}{dz} = -\log(z-a).$$

Thus, in order to discover the function of a complex variable which gives rise to the two systems of confocal conics, we have to solve the equation

$$\log\frac{dw}{dz} = -\frac{1}{2}\log(z-a) - \frac{1}{2}\log(z+a).$$

This gives us $w = \cosh^{-1}(z/a)$, *i.e.* $z = a\cosh w$; and, if we turn the plane of w through a right angle by writing iw for w, we arrive at the well-known form

$$z = a\cos w.$$



Let C be the centre of a circle, P a point on the circle, and S and S' two inverse points with respect to the circle. Then, taking the above figure, we have the angular relation

$$PCA = PSC + CPS = PSC + PS'C.$$

Further, if we draw two circles passing through P and having their centres at S and S', the tangents at P to the three circles in the figure will be respectively perpendicular to SP, $S'P$ and CP. Thus the relation between the angles which these three tangents make with the line $S'SA$ is exactly similar to that given above. Thus we see that a system of coaxal circles, having S and S' for limiting points, will form an equipotential system, being derivable with the aid of our theorem from two systems of concentric circles having S and S' for centres. To obtain the function of a complex variable from which this system is derivable, we have to solve the equation

$$\log\frac{dw}{dz} = -\log(z-a) - \log(z+a) + \log k,$$

where a and k are real quantities. This gives us

$$w = \frac{k}{2a}\log\frac{z-a}{z+a}.$$

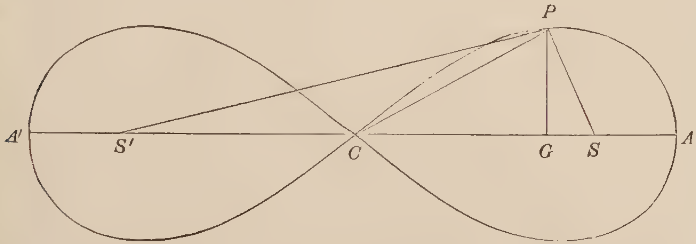
We will now make $k = a$, so that this becomes

$$2w = \log\frac{z-a}{z+a},$$

which may be written in the equivalent form $z = -a\coth w$. And, if we turn the plane of w and the plane of z each through a right angle by writing iw for w and iz for z, we have the form

$$z = a\cot w.$$

Next suppose that we have a lemniscate, having S and S' for foci, and C for node. Let P be a point on the curve, and join CP, SP and $S'P$. Also, let PT and PG be the tangent and normal at P. Then, referring to the figure below, we have the angle $SPG = CPS'$, and therefore



$$\begin{aligned} PSA + PS'A - PCA &= SPC + PCA - CPS' \\ &= CPG + PCA \\ &= PGA. \end{aligned}$$

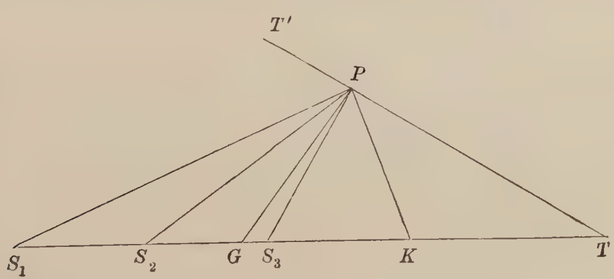
It is evident that a relation exactly similar to this will be satisfied by the angles which PT and the tangents at P to the circles passing through that point and having S', C, S for centres, make with the line $S'SA$. Thus from the three systems of concentric circles having S', C, S for centres we can deduce a system of equipotential curves having the lemniscate for one of their number. To do this we have to solve the equation

$$\log \frac{dw}{dz} = -\log(z-a) - \log(z+a) + \log z + \log k,$$

where k and a are real quantities. This gives us

$$w = \frac{k}{2} \log(z^2 - a^2).$$

As a final example of this character, we will discuss the Cartesian Oval. Prof. Crofton gives the following theorem:



The arc of a Cartesian Oval makes equal angles with the right line drawn from the point to any focus and the circular arc drawn from it through the other two foci.

Taking the above figure, let S_1, S_2, S_3 be the three foci of the curve, P a point on the curve, and PT and PG the tangent and normal at P. Also, let PK be the tangent at P to the circle passing round the triangle S_2PS_3. Then by Crofton's theorem we have the angle $TPK = S_1PT'$, and therefore $S_1PG = KPG$. And we have $KPG = S_3PK + GPS_3 = PS_2K + PS_3K - PGK$. Also $S_1PG = PGK - PS_1K$, and thus we obtain the angular relation

$$2 . PGK = PS_1K + PS_2K + PS_3K.$$

Hence we see that a system of confocal Cartesians form an equipotential system; and, taking S_1 for origin, and writing $S_1S_2 = a$ and $S_1S_3 = b$, we see that the requisite function of a complex variable is obtained by solving the equation

$$\log \frac{dw}{dz} = -\tfrac{1}{2}\log z - \tfrac{1}{2}\log(z-a) - \tfrac{1}{2}\log(z-b) + \log f.$$

Thus we have
$$w = \int_0^z \frac{fdz}{\sqrt{z(z-a)(z-b)}},$$

which, if we write $f = \sqrt{b}/2$, is equivalent to $z = a \operatorname{sn}^2 w$, $k = \sqrt{a}/\sqrt{b}$.

3. The fact that $\log h - i\vartheta$ is expressible as a function of a complex variable is well known, but it does not seem to have been noticed that it is only the first member of a whole series of expressions of the same type. It is obvious that the operation that was performed upon w to obtain $\log h - i\vartheta$ may be repeated, and thus we shall be furnished with a new expression of a general type capable of being expressed as a function of a complex variable.

If ρ_1 and ρ_2 be the respective radii of curvature of the curves $\xi = \text{const.}$ and $\eta = \text{const.}$ at their point of intersection, then

$$\frac{1}{\rho_1} = -\frac{\partial h}{\partial \xi}, \text{ and } \frac{1}{\rho_2} = -\frac{\partial h}{\partial \eta}.$$

Thus we have

$$\left(\frac{\partial \log h}{\partial x}\right)^2 + \left(\frac{\partial \log h}{\partial y}\right)^2 = \frac{1}{h^2}\left\{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right\}$$
$$= \left(\frac{\partial h}{\partial \xi}\right)^2 + \left(\frac{\partial h}{\partial \eta}\right)^2 = \left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2.$$

Further

$$\tan^{-1}\left\{\frac{\partial \log h}{\partial y}\Big/\frac{\partial \log h}{\partial x}\right\} = \tan^{-1}\left\{\frac{\partial h}{\partial y}\Big/\frac{\partial h}{\partial x}\right\}$$

$$= \tan^{-1}\frac{-\dfrac{1}{\rho_1}\dfrac{\partial \xi}{\partial y}-\dfrac{1}{\rho_2}\dfrac{\partial \eta}{\partial y}}{-\dfrac{1}{\rho_1}\dfrac{\partial \xi}{\partial x}-\dfrac{1}{\rho_2}\dfrac{\partial \eta}{\partial x}}$$

$$= \tan^{-1}\frac{\dfrac{\partial \xi}{\partial y}\Big/\dfrac{\partial \xi}{\partial x}+\dfrac{1}{\rho_2}\Big/\dfrac{1}{\rho_1}}{1-\dfrac{1}{\rho_2}\dfrac{\partial \xi}{\partial y}\Big/\dfrac{1}{\rho_1}\dfrac{\partial \xi}{\partial x}}$$

$$= \tan^{-1}\left\{\frac{\partial \xi}{\partial y}\Big/\frac{\partial \xi}{\partial x}\right\} + \tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}$$

$$= \vartheta + \tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}.$$

Thus we obtain a new function of a complex variable, viz.

$$\frac{1}{2}\log\left\{\left(\frac{1}{\rho_1}\right)^2+\left(\frac{1}{\rho_2}\right)^2\right\} - i\left[\vartheta + \tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}\right],$$

or, as we may write it, $\log h' - i\vartheta'$. Subtracting $\log h - i\vartheta$ we see that

$$\log\frac{h'}{h} - i(\vartheta' - \vartheta)$$

or

$$\log\frac{h'}{h} - i\tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}$$

is also expressible in the form of a function of a complex variable. The appearance of h in the latter of these two functions and that of ϑ in the former, just preclude the possibility of using either of them for the purpose of modifying, so as to suit cases in which the boundaries are circular, the method adopted by Kirchhoff for discovering solutions in cases where the boundaries are straight.

It is evident that the process we have employed could be repeated again and again, and thus we should be furnished with a whole series of expressions of a general form capable of being expressed as functions of a complex variable. These expressions, however, rapidly become more complicated. If we pursue the investigation one step further, we obtain an expression $\log h'' - i\vartheta''$, where

$$h''^2 h'^2 = \left\{h\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_2}\right) - \frac{1}{\rho_1\rho_2}\right\}^2 + \left\{h\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_1}\right) + \left(\frac{1}{\rho_2}\right)^2\right\}^2$$

$$= \left\{h\frac{\partial}{\partial\eta}\left(\frac{1}{\rho_1}\right) - \frac{1}{\rho_1\rho_2}\right\}^2 + \left\{h\frac{\partial}{\partial\eta}\left(\frac{1}{\rho_2}\right) + \left(\frac{1}{\rho_1}\right)^2\right\}^2,$$

and

$$\vartheta'' = 2\vartheta - \vartheta' + \tan^{-1}\frac{h\dfrac{\partial}{\partial\xi}\left(\dfrac{1}{\rho_2}\right) - \dfrac{1}{\rho_1\rho_2}}{h\dfrac{\partial}{\partial\xi}\left(\dfrac{1}{\rho_1}\right) + \left(\dfrac{1}{\rho_2}\right)^2}$$

$$= 2\vartheta - \vartheta' - \tan^{-1}\frac{h\dfrac{\partial}{\partial\eta}\left(\dfrac{1}{\rho_1}\right) - \dfrac{1}{\rho_1\rho_2}}{h\dfrac{\partial}{\partial\eta}\left(\dfrac{1}{\rho_2}\right) + \left(\dfrac{1}{\rho_1}\right)^2}.$$

4. Since $\log h - i\vartheta$ is expressible as a function of a complex variable, it follows that

$$\frac{\partial^2 \log h}{\partial\xi^2} + \frac{\partial^2 \log h}{\partial\eta^2} = 0$$

and

$$\frac{\partial^2\vartheta}{\partial\xi^2} + \frac{\partial^2\vartheta}{\partial\eta^2} = 0.$$

The first of these equations is equivalent to

$$\frac{\partial}{\partial\xi}\left(\frac{1}{h\rho_1}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{h\rho_2}\right) = 0,$$

i.e. to

$$h\left\{\frac{\partial}{\partial\xi}\left(\frac{1}{\rho_1}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\rho_2}\right)\right\} + \left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2 = 0,$$

which is the form assumed by Lamé's equation connecting the curvatures of the two families of a system of orthogonal curves, when the system is derivable from a function of a complex variable. It would be an interesting subject for research to discover whether there be an analogue to the equation

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) . \log\left\{\left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2\right\} = 0$$

in the general theory of orthogonal curves.

Since $\dfrac{\partial(-\vartheta)}{\partial\xi} = -\dfrac{\partial \log h}{\partial\eta}$ and $\dfrac{\partial(-\vartheta)}{\partial\eta} = \dfrac{\partial \log h}{\partial\xi}$,

therefore we have

$$\frac{\partial\vartheta}{\partial\xi} = -\frac{1}{h\rho_2} \text{ and } \frac{\partial\vartheta}{\partial\eta} = \frac{1}{h\rho_1}.$$

Thus the second of our two equations becomes

$$0 = -\frac{\partial}{\partial \xi}\left(\frac{1}{h\rho_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h\rho_1}\right)$$
$$= -\frac{1}{h}\left\{\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_2}\right) - \frac{\partial}{\partial \eta}\left(\frac{1}{\rho_1}\right)\right\} - \frac{1}{h^2}\left\{\frac{1}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\right\},$$

and therefore

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_2}\right) = \frac{\partial}{\partial \eta}\left(\frac{1}{\rho_1}\right),$$

as is obvious from the expressions for ρ_1 and ρ_2.

From the modified form of Lamé's equation given above we deduce

$$\log\left\{\left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2\right\} = \log h + \log\left\{\frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2}\right\};$$

therefore

$$2(\log h' - i\vartheta') = \log h + \log\left\{\frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2}\right\} - 2i\left[\vartheta + \tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}\right],$$

which shews that

$$\log\left\{\frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2}\right\} - i\left[\vartheta + 2\tan^{-1}\left\{\frac{1}{\rho_2}\Big/\frac{1}{\rho_1}\right\}\right]$$

is expressible as a function of a complex variable. Thus we have

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) . \log\left\{\frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2}\right\} = 0,$$

from which we easily deduce

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) . \log\left\{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right\} = 0.$$

If we expand the equation

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) . \log\left\{\left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2\right\} = 0,$$

take logarithms, and make use of all of the expressions in Art. 3, we deduce that

$$\log\left[\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \cdot \frac{1}{4}\left\{\left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2\right\}\right]$$
$$- 2i\left[\vartheta + \tan^{-1}\frac{h\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_2}\right) - \frac{1}{\rho_1\rho_2}}{h\frac{\partial}{\partial \xi}\left(\frac{1}{\rho_1}\right) + \left(\frac{1}{\rho_2}\right)^2}\right]$$

is expressible as a function of a complex variable. Hence we see that

$$\log\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\cdot\left\{\left(\frac{1}{\rho_1}\right)^2+\left(\frac{1}{\rho_2}\right)^2\right\}\right]$$

is also a solution of Laplace's equation.

It seems highly probable that this process may be repeated again and again, and thus we should be furnished with a second series of expressions of a general type capable of being expressed as functions of a complex variable.

The modified form of Lamé's equation given in this article was used in reducing the last expression of Art. 3.

5. Suppose that we are considering a case of irrotational motion of an incompressible fluid. Let p be the pressure and h the velocity at any point of the fluid, ξ the velocity potential, η the current function, V the potential of the external forces, and δ the density of the fluid. We have the equation

$$\frac{p}{\delta}+V+\frac{\partial\xi}{\partial t}+\frac{1}{2}h^2=f(t),$$

from which we easily deduce

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left\{\frac{p}{\delta}+V+\frac{\partial\xi}{\partial t}\right\}-h\left\{\frac{\partial}{\partial\xi}\left(\frac{1}{\rho_1}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{\rho_2}\right)\right\}+\left(\frac{1}{\rho_1}\right)^2+\left(\frac{1}{\rho_2}\right)^2=0,$$

where ρ_1 and ρ_2 are the respective radii of curvature of the curves $\xi=\text{const.}$ and $\eta=\text{const.}$ at their point of intersection. Making use of the modified form of Lamé's equation, we deduce

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left\{\frac{p}{\delta}+V+\frac{\partial\xi}{\partial t}\right\}+2\left\{\left(\frac{1}{\rho_1}\right)^2+\left(\frac{1}{\rho_2}\right)^2\right\}=0.$$

If the fluid be homogeneous, and V satisfy Laplace's equation, this becomes

$$\frac{1}{\delta}\left\{\frac{\partial^2 p}{\partial\xi^2}+\frac{\partial^2 p}{\partial\eta^2}\right\}+2\left\{\left(\frac{1}{\rho_1}\right)^2+\left(\frac{1}{\rho_2}\right)^2\right\}=0.$$

From this it follows that

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left\{\log\left(\frac{\partial^2 p}{\partial\xi^2}+\frac{\partial^2 p}{\partial\eta^2}\right)\right\}=0,$$

from which we easily deduce

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left\{\log\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\right\}=0.$$

The similarity of this equation to the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).\log\left\{\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right\}=0$$

given above, indicates the possibility of deriving solutions suitable for free surfaces from known solutions applying to cases in which the boundaries are fixed*.

6. It is always possible to find an indefinite number of systems of equipotential lines each having a given curve for one of the curves of the system. We have only to express the coordinates of a point on the curve in terms of a single parameter, which may be done in an indefinite number of ways. Thus, suppose that we have $x=\phi(\alpha)$ and $y=\psi(\alpha)$; then

$$z=x+iy=\phi(\alpha)+i\psi(\alpha).$$

Now write

$$z=\phi(\xi+i\eta)+i\psi(\xi+i\eta).$$

This will give us two systems of equipotential curves $\xi=$ const. and $\eta=$ const., the series $\eta=$ const. containing the given curve, since that curve corresponds to the value $\eta=0$.

Further, if we have two or more systems of equipotential curves, each containing a given curve, we may combine them with the aid of the theorems contained in Art. 1 and deduce new systems containing the said curve.

In a large number of physical problems, however, we find that, as a first step to the solution of the problem, we have to discover a system of equipotential curves which contains two given curves; and, although we can write down the functions of a complex variable corresponding to as many systems as we please that contain either of the two curves, yet we shall often find it impossible to hit upon that particular one which corresponds to the system containing both the curves. What is greatly to be desired is the discovery of a synthetic method that would enable us to obtain the equipotential system containing two given curves. This is, of course, a very hard problem, but I am inclined to think that much light might be thrown upon it by the study of general forms like those given in Articles 3 and 4. The only methods of a synthetic character that we now possess are the method of Kirchhoff, alluded to above, and the method of images. The method of Kirchhoff, however, is very limited in its application, and the method of images becomes in many cases too cumbrous for use.

* We have to take a value of h, derived from some known case of fluid motion, and split it into two parts such that one part and the logarithm of the other part each satisfy Laplace's equation.

We may remark, however, that if by any means we can discover the distribution of ξ along one of the bounding curves, supposing those curves to correspond to two values of η, we have virtually solved the problem. We shall be able to express the coordinates of any point on the bounding curve in terms of the value of ξ at that point, and thus we shall obtain an equation of the form $z=f(\xi)$ which must hold along the boundary. The function of a complex variable that we are in search of will then be given by the equation $z=f(w)$.

7. There is one more point to which I wish to draw attention. In a paper in the fifty-fifth volume of *Crelle* entitled "Ueber die graphische Darstellung imaginärer Funktionen", Siebeck gives the following theorem:

If two diagrams are such that one may be derived from the other by means of an isogonal transformation, then to a system of straight lines in the one diagram there will correspond a system of confocal curves in the other.

If the transformation be derivable from the equation $z=f(w)$, the straight lines lying in the w plane and the curves in the z plane, then Siebeck appears to prove that the positions of the foci of the system of curves in question are obtained by substituting the roots of the equation $f'(w)=0$ for w in the expression $f(w)$. Now, by choosing the proper transformation, we can make a given system of equipotential curves in the z plane correspond to a system of parallel straight lines in the w plane. Thus Siebeck's theorem would seem to be equivalent to the statement that all equipotential systems are confocal. This is obviously untrue, since the system of coaxal circles discussed above is not a confocal system. In fact, if we apply Siebeck's method of finding foci to this case, we find that it gives the limiting points of the system which are obviously not foci*. Thus Siebeck's statement of the theorem is not quite correct as it stands. It is true that the points obtained by putting the roots of $f'(w)=0$ for w in the expression $f(w)$, stand in the same relationship to all the members of the equipotential system, but these points are not always foci, though in a great many cases they are. It is, however, highly probable that the majority of, perhaps all, confocal systems are also equipotential systems, it being understood that by confocal systems we mean systems the members of which have all their foci fixed including those at infinity.

In order to illustrate this matter further, consider the trans-

* In this case, also, Siebeck's method fails to give the positions of the foci of the constituent members of the system.

formation $z = a \operatorname{cn} w + ib \operatorname{sn} w$. In this case the curve $\eta = 0$ is an ellipse having its semi-axes of lengths a and b; and we have

$$\frac{dz}{dw} = (-a \operatorname{sn} w + ib \operatorname{cn} w) \operatorname{dn} w.$$

Substituting the roots of $f'(w) = 0$ for w in the expression $f(w)$ we obtain the six points

$$\pm\sqrt{a^2 - b^2}, \quad \pm\frac{i}{k}(ak' + b), \quad \pm\frac{i}{k}(ak' - b).$$

The first pair of points contains the two foci of the ellipse, and although the other four points may be foci of the other members of the system, they are obviously not foci of the ellipse.

(3) *On the Stability of Elastic Systems.* By G. H. Bryan, B.A., St Peter's College.

1. Kirchhoff was the first to shew* that if we are given the bodily forces acting on an elastic solid, and are also given either the surface tractions or surface displacements, there is one and only one state of strain in which the body can be in equilibrium, and that equilibrium is essentially stable for all displacements with the exception of rigid-body displacements.

Euler found that a thin wire or shaft of length l and flexural rigidity EI becomes unstable if the thrust applied to its ends be greater than that given by the formula

$$\frac{P}{EI} = \frac{\pi^2}{l^2}.$$

Greenhill† has worked out the corresponding formulae when the wire transmits both end thrust and couple and is also supposed to be under the influence of "centrifugal force." He has likewise determined‡ the greatest height of a thin vertical pole or tree consistent with stability, the diameter being a known function of the height.

These appear to be the only instances in which the question of stability has been discussed in connection with the theory of Elasticity. It therefore appeared to me that it would be worth while to give a general investigation of the circumstances under which an elastic system can be in unstable equilibrium for other than rigid-body displacements of the various bodies forming the system. I have shewn that, in general, the only systems which

* *Vorlesungen über Math. Physik.* 27, § 2.

† *Proc. Institution of Mechanical Engineers*, April, 1883, p. 182.

‡ *Camb. Phil. Proceedings*, Vol. IV. p. 66.

are not essentially stable, except for displacements differing infinitely little from those of a system of rigid bodies are such thin wires, plates, or shells as are capable of being deformed by pure bending or twisting.

It is to be remembered that the strains which most substances are capable of undergoing without losing their elastic properties must be confined within certain extremely small limits, it is only such imperfectly elastic substances as jelly or indiarubber, which can be subjected to finite strain without breaking. We shall therefore usually suppose the limits of the elasticity to be small, the elastic constants being large quantities so that the limits of the stresses are finite.

2. Let an elastic solid be in equilibrium in a state of strain under any system of external forces and constraints, the displacements of any point being u, v, w and the strains being e, f, g, a, b, c as usual. Let ϕ be the elastic potential or potential energy of strain per unit volume, so that for an isotropic body

$$\phi(e, f, g, a, b, c) \equiv \tfrac{1}{2}(m+n)(e+f+g)^2 \\ + \tfrac{1}{2}n\,(a^2+b^2+c^2-4fg-4ge-4ef) \ldots\ldots\ldots (1)^*.$$

Let the potential of the bodily forces at the point $(x+u,\ y+v,\ z+w)$ be V and let ΥdS be the potential energy of the surface tractions on the surface element dS. The whole potential energy of the system in the position of equilibrium will therefore be

$$W = \iiint \phi\, dx dy dz + \iiint \rho V dx dy dz + \iint \Upsilon dS \ldots\ldots (2),$$

if we suppose the bodily forces to be due to external causes, not to self-gravitation of the solid.

Equilibrium will be stable if W be a true minimum, unstable if W be a maximum or minimax. Let the displacements of any point receive small variations δu, δv, δw. The condition of equilibrium gives the ordinary variational equation

$$\delta W = \iiint \delta\phi\, dx dy dz + \iiint \rho\, \delta V dx dy dz + \iint \delta \Upsilon dS = 0 \ldots (3).$$

Proceeding to the second variations we see that the condition that equilibrium may be unstable requires that for *some* variations

$$\delta^2 W = \iiint \delta^2\phi\, dx dy dz + \iiint \rho\delta^2 V dx dy dz + \iint \delta^2 \Upsilon dS < 0 \ldots (4).$$

* Employing the elastic constants of Thomson and Tait.

Employing as usual P, Q, R, S, T, U to denote the stress components in the position of equilibrium, X, Y, Z the bodily forces, F, G, H the surface tractions

$$\delta\phi = P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c \text{ ....(5)},$$

and since $\phi(e, f, g, a, b, c)$ is a homogeneous quadratic function of e, f, g, a, b, c

$$\begin{aligned}\delta^2\phi &= \delta P\delta e + \delta Q\delta f + \delta R\delta g + \delta S\delta a + \delta T\delta b + \delta U\delta c \\ &= 2\phi(\delta e, \delta f, \delta g, \delta a, \delta b, \delta c) \text{ .... (6)},\end{aligned}$$

and is therefore, we know, essentially positive. Since V is a function of $x+u$, $y+v$, $z+w$, therefore

$$\delta V = \delta u\frac{\partial V}{\partial x} + \delta v\frac{\partial V}{\partial y} + \delta w\frac{\partial V}{\partial z} = -X\delta u - Y\delta v - Z\delta w \text{... (7)},$$

$$\begin{aligned}\delta^2 V = \delta u^2\frac{\partial^2 V}{\partial x^2} + \delta v^2\frac{\partial^2 V}{\partial y^2} + \delta w^2\frac{\partial^2 V}{\partial z^2} \\ + 2\delta v\,\delta w\frac{\partial^2 V}{\partial y\partial z} + 2\delta w\,\delta u\frac{\partial^2 V}{\partial z\partial x} + 2\delta u\,\delta v\frac{\partial^2 V}{\partial x\partial y} \\ = -\delta X\delta u - \delta Y\delta v - \delta Z\delta w \text{ ....(8)},\end{aligned}$$

lastly,
$$\delta\Upsilon = -F\delta u - G\delta v - H\delta w \text{.....................(9)},$$

$$\delta^2\Upsilon = -\delta F\delta u - \delta G\delta u - \delta H\delta w \text{ ....(10)}.$$

3. Since $\delta^2\phi$ is essentially positive the inequality (4) can only hold if the sum of the second and third terms is negative and numerically greater than the first, and therefore *à fortiori* cannot hold if they are small compared with the first. From this we may shew that the displacement must in general be such that the strain variations (δe, δf, δg, δa, δb, δc) are infinitely small compared with the displacement variations (δu, δv, δw).

For suppose these variations to be small quantities of the same order. Then in order that $\iiint\rho\delta^2 V dxdydz$ may be comparable in magnitude with $\iiint\delta^2\phi\, dxdydz$, $\frac{\partial^2 V}{\partial x^2}$, ... $\frac{\partial^2 V}{\partial y\partial z}$, ... must be of magnitude comparable with the elastic constants m, n, and this must necessarily also be the case with the forces $-\frac{\partial V}{\partial x}$, $-\frac{\partial V}{\partial y}$, $-\frac{\partial V}{\partial z}$, as appears at once by integration. But from the equation of equilibrium (3), forces of this magnitude will produce finite strains in the body instead of infinitely small ones. As above stated, this is possible only for a few very extensible substances like jelly.

Thus the paraboloidal jelly discussed by Greenhill* may sometimes be unstable under its own weight, even if its transverse section is so large that his mode of treatment (in which the formulae for the bending moment of a thin wire whose middle line is the axis of the paraboloid are employed) gives only very roughly approximate results.

But these cases are of little interest. In the solids with which we are dealing, such finite strains cannot exist without "set" being caused, and therefore the second variation of the potential energy of the forces must be small compared with that due to strain.

If the body is self-attracting the same thing is equally true. For the potential energy of self-attraction is

$$\iiint\iiint \frac{\rho\, dx\, dy\, dz\, .\, \rho' dx' dy' dz'}{R},$$

where

$$R^2 = (x + u - x' - u')^2 + (y + v - y' - u')^2 + (z + w - z' - w')^2,$$

and the integral extends to every pair of elements of the body. The first and second variations of this are

$$\iiiint\iint \rho\, dx\, dy\, dz\, \rho' dx' dy' dz'$$
$$\left[(\delta u - \delta u')\frac{\partial}{\partial x}\left(\frac{1}{R}\right) + (\delta v - \delta v')\frac{\partial}{\partial y}\left(\frac{1}{R}\right) + (\delta w - \delta w')\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right],$$

and

$$\iiint\iiint \rho\, dx\, dy\, dz\, \rho' dx' dy' dz'$$
$$\left\{(\delta u - \delta u')\frac{\partial}{\partial x} + (\delta v - \delta v')\frac{\partial}{\partial y} + (\delta w - \delta w')\frac{\partial}{\partial z}\right\}^2 \frac{1}{R}$$

respectively.

By the variational equation of equilibrium $\rho\rho'$ or ρ^2 must be comparable with the stresses in the position of equilibrium and therefore in general small compared with the elastic constants. Hence, just as in the previous case, the second variation of the potential energy of self-attraction cannot in general be made comparable with that due to strain, except by taking the displacement variations to be such that the strain variations are small in comparison.

Next let us examine whether the surface integral $\iint \delta^2 \Upsilon dS$, which is the second variation of the energy of the surface tractions

* *Camb. Phil. Proc.* Vol. IV. *loc. cit.*

can be comparable with $\iiint \delta^2\phi\, dx dy dz$. Writing these terms in the forms $\iint (\delta F \delta u + \delta G \delta v + \delta H \delta w) dS$ and

$$\iiint (\delta P \delta e + \delta Q \delta f + \delta R \delta g + \delta S \delta a + \delta T \delta b + \delta U \delta c)\, dx dy dz,$$

respectively, we see, that if the variations of the strains are still supposed to be comparable with those of the surface displacements, the variations δF, δG, δH of the surface tractions must be comparable with the stress variations $\delta P, \delta Q, \delta R, \delta S, \delta T, \delta U$, a condition which is otherwise almost evident. Now this will be the case if the surface tractions are due to the reactions of other elastic solids with which the body under consideration is in contact. The same is equally true if they are produced by the pressure of liquids whose elasticity of volume is comparable with that of the solid, provided the variational displacement involves changes in the volume of the liquid. In such instances let us consider the potential energy of the whole system instead of that of a single body. Then the total potential energy due to work done by the actions and reactions of the various bodies of the system vanishes identically by Newton's third law, and therefore also its successive variations vanish identically. On the other hand, the second variations of the potential energies of strain of all the bodies are essentially positive and therefore that of the whole system is essentially positive. Hence the whole system will in general be stable, and therefore any body of the system will necessarily *à fortiori* be in stable equilibrium for displacements of the kind here considered. In every other conceivable case, δF, δG, δH will be linear functions of δu, δv, δw and their differential coefficients, in which the coefficients are quantities of the same order of magnitude as the surface tractions F, G, H in the position of equilibrium, and are therefore small compared with the elastic constants, hence δF, δG, δH are small compared with δP, δQ, δR, δS, δT, δU, and $\iint \delta^2 \Upsilon\, dS$ is small compared with $\iiint \delta^2\phi\, dx dy dz$.

4. From these discussions it follows that the equilibrium of an elastic solid acted on by any system of bodily forces or surface tractions, which produce only small strains of the substance, is essentially stable for all displacements with the exception of those in which the variations produced in the strains are very small in comparison with the variations in the positions of the particles of the solid.

Now if the strains were zero the displacement would be purely that of a rigid body. If they are infinitely small compared with

the displacements and all the dimensions of the body are finite and comparable, it may readily be proved that the displacements must differ from rigid-body displacements by terms that are infinitely small in comparison, (although this is not necessarily true for a body whose dimensions are not all finite and comparable). If therefore we suppose the portions of δu, δv, δw due to the rigid-body displacement alone to exist, the difference produced in the second and third terms of the inequality (4) will in general be infinitely small, while the first term will be zero instead of being positive, and the body must therefore *à fortiori* be unstable for the rigid-body displacement alone.

We may however have systems of finite bodies under tight-fitting constraints which allow of nearly, but not quite, rigid-body displacements or in which the potential energy of the forces differs considerably for small deviations from rigid-body displacements, in such cases unstable equilibrium may be broken by displacements accompanied by small variations of the strains, but these cases present no points of interest. A smooth sphere which, if slightly squeezed, can be pushed into the aperture in a solid anchor ring and fits it tightly is an illustration of such an unstable system.

It is to be mentioned that in Kirchhoff's investigation the variations of the surface tractions and bodily forces are supposed to be zero, while if the surface displacements are given their variations are zero. Hence the second variation of the potential energy is entirely due to strain. I have taken account of these variations of the impressed forces and discussed under what circumstances alone their effect becomes important.

Stability of Wires, Plates and Thin Shells.

5. At the commencement of this paper I alluded to Euler's and Greenhill's determination of the criteria of instability of a thin wire. In like manner we may find the condition that a thin plate subject to given edge thrusts in its plane may be unstable. At present we shall content ourselves with the simple example of an infinitely long strip of breadth l and thickness $2h$ acted on by normal edge thrust in its plane and of magnitude P per unit of length of the edge. In this example, which closely resembles Euler's problem of the wire with end thrust, the condition of instability is

$$P > \tfrac{8}{3} n h^3 \left(\frac{m}{m+n}\right) \frac{\pi^2}{l^2} \qquad \text{.........................(11)},$$

provided that, either the edges are fixed in position or the tangent planes along them are fixed in direction. This gives a very rough

idea of the greatest thrust which can be applied along the ends of a very broad thin flat piece of clock spring, without causing it to double up.

To prove the above result take the axis of y along one edge of the strip, the axis of x being in its plane perpendicular to the edge, and that of z perpendicular to the surface. Let the plate be deformed so that any point of it receives a small displacement w, independent of y, perpendicular to the plane of the plate. Then if s is the length measured along the new middle surface of the plate in the plane of xz, the work done in stretching the surface by the thrust P per unit length of the strip is*

$$P\int_0^l \left(\frac{ds}{dx}-1\right) dx = \tfrac{1}{2}P\int_0^l \overline{\frac{dw}{dx}}\Big|^2 dx,$$

while the potential energy of bending per unit length is

$$\tfrac{4}{3}nh^3\left(\frac{m}{m+n}\right)\int_0^l \overline{\frac{d^2w}{dx^2}}\Big|^2 dx.$$

Hence the plate will be stable if

$$\tfrac{1}{2}P\int_0^l \overline{\frac{dw}{dx}}\Big|^2 dx < \tfrac{4}{3}nh^3\left(\frac{m}{m+n}\right)\int_0^l \overline{\frac{d^2w}{dx^2}}\Big|^2 dx \quad\text{.........(12)}$$

for every possible deformation.

If the edges are fixed in position $w=0$ both when $x=0$ and when $x=l$, and therefore w may be expanded by Fourier's series in the form

$$w = \Sigma_{r=1}^{r=\infty} a_r \sin\frac{r\pi x}{l} \quad\text{......................(13).}$$

If on the other hand the directions of the tangent planes along the edges are fixed, $\dfrac{dw}{dx}=0$ at either end, and w may be expanded in the form

$$w = \Sigma_{r=0}^{r=\infty} a_r \cos\frac{r\pi x}{l} \quad\text{......................(14).}$$

In either case the condition for stability (12) requires that

$$\tfrac{1}{2}P\Sigma_{r=1}^{r=\infty} a_r^2\frac{r^2\pi^2}{l^2} < \tfrac{4}{3}nh^3\left(\frac{m}{m+n}\right)\Sigma_{r=1}^{r=\infty} a_r^2\frac{r^4\pi^4}{l^4},$$

or

$$P < \tfrac{8}{3}nh^3\left(\frac{m}{m+n}\right)\frac{\pi^2}{l^2}\frac{\Sigma_{r=1}^{r=\infty} r^4 a_r^2}{\Sigma_{r=1}^{r=\infty} r^2 a_r^2} \quad\text{..................(15)}$$

for all values of the constants a_r.

* Compare Lord Rayleigh, *Sound*, Vol. I. p. 136.

Now $\Sigma r^4 a_r^2 / \Sigma r^2 a_r^2$ will have a minimum value $= 1$ when all the constants $a_2 a_3 \ldots$ vanish, and a_1 is not $= 0$. Hence the plane form will be stable or unstable, according as P is less or greater than

$$\tfrac{8}{3} nh^3 \left(\frac{m}{m+n}\right) \frac{\pi^2}{l^2}.$$

If equilibrium in the plane form is critical so that P is equal to the above value, the pressure per unit of surface over the edge is p where

$$p = \frac{P}{2h} = \tfrac{4}{3} nh^2 \left(\frac{m}{m+n}\right) \frac{\pi^2}{l^2} \quad \text{.....................(16)},$$

and if σ is the measure of the compression of the surface

$$8nh \left(\frac{m}{m+n}\right) \sigma = P;$$

$$\therefore \quad \sigma = \frac{\pi^2}{3} \frac{h^2}{l^2} \quad \text{.........................(17)}.$$

In the wire problem of Euler

$$p = uk^2 . \left(\frac{3m-n}{m}\right) \frac{\pi^2}{l^2} \quad \text{.....................(18)}.$$

$$\sigma = \pi^2 \frac{k^2}{l^2} \quad \text{.............................(19)},$$

where k is the radius of gyration of the cross section of the wire about the line about which bending takes place.

The values of σ will therefore be equal in the respective cases if

$$k^2 = \frac{h^2}{3} \quad \text{.............................(20)}.$$

By taking h and k sufficiently small compared with l these values of σ may be made as small as we please, and therefore the lamina, as well as the wire, will, if thin enough, be unstable for thrusts far less than those required to produce "set."

6. We shall now explain the possible instability of such thin solids from general observations.

We may point out that the surface integrals both in δW, $\delta^2 W$ increase indefinitely in importance compared with the volume integrals, if one or two of the dimensions of the body become indefinitely diminished, and this is in accordance with the known fact that the surface tractions on an infinitely thin wire or shell must be infinitely small. Hence, although the ratios of $\delta F, \delta G, \delta H$ to the displacement variations, when (4) is satisfied, diminish indefinitely, yet if we suppose these ratios to be always comparable

with F, G, H, the latter will still be so large as to produce "set," unless the strain variations be infinitely small in comparison with those of the displacements.

But this is precisely what happens when such a thin wire or shell is deformed by pure bending or twist.

In Kirchhoff's theory of the bent wire*, the wire is supposed divided up into a number of small elements by planes perpendicular to the middle line at distances apart of order ϵ, where ϵ is comparable with the thickness of the wire; ξ, η, ζ are supposed to be the coordinates of the centre of any such element. With this centre as origin, Kirchhoff takes a system of rectangular axes (x, y, z), that of z being the tangent to the middle line, and that of x being a fixed transverse of the wire. These axes are supposed to move with the element. He also introduces three quantities p, q, r, depending on the bending and twist, which we shall denote by κ, λ, τ. If the wire is but slightly bent κ, λ, τ represent the rate of change, per unit length of the middle line, of the angles through which the axes in the various elements are rotated by bending. Supposing θ to be a quantity of the same order of magnitude as κ, λ, τ, then by considering a finite length of wire, it is evident that the component rotations of the axes in any element, as well as the component translations of its centre (ξ, η, ζ), are quantities of order θ. The total displacement of any point in the element is therefore compounded of a rigid-body displacement of order θ, together with the displacement of the point relative to the moving axes of x, y, z.

Now if we suppose that instead of the wire being purely bent the extension of the mean fibre is σ, the expressions for these small relative displacements are to a first approximation

$$\left.\begin{aligned} u &= -\sigma\mu x + \tau yz - \lambda\frac{z^2}{2} - \lambda\mu\frac{x^2 - y^2}{2} + \kappa\mu xy \\ v &= -\sigma\mu y - \tau zx + \kappa\frac{z^2}{2} + \kappa\mu\frac{x^2 - y^2}{2} - \lambda\mu xy \\ w &= \sigma z + \tau w_1 + \lambda xz - \kappa yz \end{aligned}\right\}\ldots\ldots(21),$$

where μ = ratio of lateral contraction to longitudinal elongation, and w_1 being found, as in Saint-Venant's problem, is a small quantity of order ϵ^2.

If $\sigma = 0$, or likewise if σ is a small quantity of order ϵ in comparison with κ, λ, τ, the expressions will be quantities of order $\epsilon^2\theta$, the strains therefore will be of the order $\epsilon\theta$, and the displacement

* *Vorlesungen*, No. 28.

of any such small element will differ from a rigid-body displacement by components which are small quantities of order ϵ in comparison. If the wire be taken sufficiently thin these terms may be made as small as we please in proportion, and a displacement, differing infinitely little from one of pure bending or twist, will therefore be exactly that kind of displacement for which alone, an element, all of whose dimensions are comparable, may be in unstable equilibrium.

The same thing is true for a thin plate or shell. We suppose such a plate divided into elementary parallelopipeds whose length and breadth are both comparable with the thickness ($2h$) and the relative displacements in such an element are referred to certain rectangular axes which move with the element*. The bending being determined by the quantities κ_1, λ_1, κ_2, if these be supposed of order of magnitude represented by θ, the components of translation of the centre of any element as well as the rotations of the axes of x, y, z in that element will be quantities of order θ. On the contrary if the displacement be one in which the middle surface is either unextended or the stretching is infinitely small in comparison with the bending, it appears just as before that the relative displacements of a point in the element referred to these moving axes are of order $h^2\theta$, the strains being of order $h\theta$. If the plate be infinitely thin, such a displacement, therefore, differs infinitely little from a rigid-body displacement, so far as any element, all of whose dimensions are comparable, is concerned.

7. It only remains for us to determine what is the order of magnitude of the small strains produced in such a thin elastic solid when the forces acting on it are so great that it is possible that equilibrium may be unstable. Let us take a thin plate both bent and stretched in the state of equilibrium so that σ_1, σ_2, ϖ are the principal extensions along two perpendicular lines in the middle surface, and the shear of the angle between these lines respectively. The total potential energy due to strain is

$$4nh\,W_2 + \tfrac{4}{3}nh^3\,W_3,$$

where†

$$W_2 = \frac{m}{m+n}\iint\{(\sigma_1+\sigma_2)^2 + 2(1-\mu)(\overline{\tfrac{1}{2}\varpi}|^2 - \sigma_1\sigma_2)\}\,dS\ldots\ldots(22).$$

$$W_3 = \frac{m}{m+n}\iint\{(\kappa_2-\lambda_1)^2 + 2(1-\mu)(\kappa_1^2 - \lambda_1\kappa_2)\}\,dS\ldots\ldots\ldots(23).$$

Let the whole potential energy of the impressed forces whether

* See Mr Love's "Note on Kirchhoff's theory of the deformation of elastic plates," *Camb. Phil. Proceedings.*

† Kirchhoff, *loc. cit.* p. 454.

bodily forces, surface tractions, edge tractions or couples round the edge be denoted by V_1. So that the total potential energy of the system is

$$W = 4nh\,W_2 + \tfrac{4}{3}nh^3 W_3 + V_1 \quad \text{...............(24)}.$$

Let the system receive a small variational displacement of order $\delta\theta$. The conditions of equilibrium (4) and instability (5) become

$$\delta W = 4nh\delta W_2 + \tfrac{4}{3}nh^3\delta W_3 + \delta V_1 = 0 \quad \text{.........(25)}.$$

$$\delta^2 W = 4nh\delta^2 W_2 + \tfrac{4}{3}nh^3\delta^2 W_3 + \delta^2 V_1 < 0 \quad \text{.........(26)},$$

of which the former must hold for all variations and the latter for *some* variations.

Now as we have already shewn $\delta^2 V$ will in general be a quantity of the same order of magnitude as $\delta V\delta\theta$. Moreover $\delta^2 W_2$, $\delta^2 W_3$ are of order $\delta\theta^2$. Suppose that equilibrium is unstable for a displacement of pure bending, then we must put $\delta\sigma_1$, $\delta\sigma_2$, $\delta\varpi$ all $=0$ and therefore $\delta^2 W_2 = 0$. In order that the inequality (26) may be satisfied $\delta^2 V_1$ must be a quantity of order $nh^3\delta\theta^2$. Hence δV is of order $nh^3\delta\theta$.

Again dW_3 involves products of κ_1, λ_1, κ_2 into their variations and is of order $\kappa_2\delta\theta$, and therefore if the plate is bent in the position of equilibrium it follows from (25) that κ_1, κ_2, λ_1 are all finite. But the strains when this is the case are small quantities of order h.

To find the strains due to extension of the middle surface of the plate in the position of equilibrium we must use the equation of virtual velocities (25) for variations of order $\delta\theta$ in which $\delta\sigma_1$, $\delta\sigma_2$, $\delta\varpi$ are not zero. We at once find that σ_1, σ_2, ϖ, the strains due to stretching, are small quantities of order h^2.

In the example we have worked out, the plate was not bent in the position of equilibrium and the result there found confirms our present general result.

In the case of the bent wire exactly the same conclusions hold good. The potential energies due to stretching and bending for an isotropic wire are

$$\tfrac{1}{2}\int E\alpha\sigma^2 ds \text{ and } \tfrac{1}{2}\int E\alpha\left(k_1^2\kappa^2 + k_2^2\lambda^2 + \frac{u}{E}k_3^2\tau^2\right)dS,$$

where $E =$ Young's modulus, $\alpha =$ area of cross section k_1, k_2 being its principal radii of gyration, k_3 a constant of the same magnitude and κ, λ are the curvatures about the *principal* axes of the section and τ the twist. Now α is of order ϵ^2, k_1, k_2, k_3 of order ϵ, ϵ being as before comparable with the thickness of the wire. It readily follows that if such a wire be unstable under any given

forces, then if these forces be such as to bend or twist the wire in the position of equilibrium, the flexion and torsion thus caused will be finite the strains being of order ϵ*. On the other hand the strains due to extension or compression of the middle line will be quantities of order ϵ^2.

From the above results we can draw the following general conclusions as to the limit of the thickness of a thin wire or plate for which unstable equilibrium is possible under any system of forces, it being remembered that the body must not be strained beyond the elastic limits.

1. If the forces are such as to produce bending in the position of equilibrium this limiting thickness is a small quantity of the same order as the total increase in length in a bar of the same material and of length equal to the greatest linear dimension of the plate or wire, when the strain is the greatest it will bear.

2. If the forces produce only extension or compression of the middle line or surface, unaccompanied by bending, the thickness may be much greater, and its limit is a small quantity of the same order as a mean proportional between the same length above mentioned and a length comparable with the greatest linear dimensions of the body.

In no case can equilibrium be unstable for displacements other than those either composed wholly of pure bending or differing infinitely little from pure flexion and torsion.

8. One consequence of this fact is worthy of notice. Gauss has proved that a thin shell in the form of a closed surface cannot be deformed by pure bending unaccompanied by extension or compression of the surface. Such a shell is, therefore, essentially stable, (although should there be a small crack or flaw in the material this would not be necessarily true).

In particular a thin hollow elastic spherical shell surrounded by a medium subjected to uniform hydrostatic pressure, will always be perfectly stable. If the pressure be sufficiently increased we shall at last arrive at a point when the material of the shell will be ruptured, or a "doke" produced when the pressure is sufficiently great to cause "set", but until this happens the shell will always continue to remain in the form of a sphere.

* This is exemplified by the condition for instability of a wire when a couple is applied at both ends, investigated by Greenhill (*loc. cit.*). When such a wire is in unstable equilibrium the measure of torsion is finite, but the shear at any point is a small quantity proportional to its distance from the axis. If the wire is of circular section the *integral twist* must exceed $2\pi(3m-n)/m$.

March 12, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

W. H. MACAULAY, M.A., Fellow of King's College, was elected a Fellow of the Society.

The following communications were made :

(1) *Note on some Experiments on the Creeping of Solutions.* By JAMES MONCKMAN (D.Sc., Lond.), Downing College.

The curious property of certain salts in solution, of covering the sides of the vessels in which they are placed with crystals is well known. It is often slow at first, as though it varied with the state of the vessel, or required something to set it off, after which it progresses rapidly. When one side of a crystal of sulphate of copper is rubbed flat and polished it is found that a solution of the same salt does not rise so high on the smooth surface as on the rough one, but that in both cases it extends further than on glass. This is shewn by placing a crystal, one of whose sides is polished, in contact with the solution in such a manner that the side may be vertical. Strips of glass are also fixed in the same liquid, and after remaining for an hour they are removed, and the portion over which the liquid has spread is measured.

An experiment gave the following numbers :—

Height to which the liquid rose on the	rough side of the crystal...	10 mm.
„ „ „	smooth „ „ ...	6 „
„ „ „	rough glass	6 „
„ „ „	smooth glass	2 „

These numbers shew that when crystals are once deposited the solution will rise more rapidly, but they afford no indication of the cause of deposition of the first line of solid. Thinking that this was probably due to the combined action of shaking and evaporation, I placed a number of watch-glasses of various sizes and curvatures upon a stone table, which was free from the vibrations of the floor. When perfectly still, sulphate of copper solution was allowed to flow into them from a burette, care being taken to prevent dropping by which waves would be produced and the watch-glass wet above the line of the liquid.

The glasses were then covered with a bell jar. After remaining thus for three days there was no trace of creeping, but when the jar was removed and evaporation allowed to commence the crystals began to appear, and, in the case of those glasses which contained sufficient liquid, the crystals spread over the edge of the vessel.

It is evident, therefore, that in order to commence in a clean smooth glass or porcelain vessel, a layer of crystals must be formed above the water-line. This is done either by the evaporation of the film left after shaking, or by the lowering of the whole surface by the same means, when a line of salt is left at the edges.

This line of salt then acts as a rough surface, and as it is not closely packed there are passages between the crystals which act as irregular capillary tubes. In the former case the thin film of liquid will soon evaporate and the creeping commence, while in the latter more time will be required. Hence the irregularity in the starting.

(2) *On the action of Acetone on the Ammonium Salts of fatty acids in presence of dehydrating agents.* By Dr RUHEMANN and D. J. CARNEGIE, B.A., Caius College.

By means of this reaction the authors obtained a base of the formula $C_9H_{15}N$ which had previously been obtained by Heintz by quite a different reaction. From their experimental results the authors conclude that this base is an unstable hydrogenised pyridine body; but their attempts to reduce it to a well-defined member of the pyridine series have so far been unsuccessful.

(3) *On the reduction of solutions of ferric salts to ferrous salts by certain metals.* By D. J. CARNEGIE, B.A., Caius College.

Certain metals (e.g. Zn, Pb, Fe, Cu) in a state of very fine division almost instantly reduce ferric salt solutions. The author develops a rapid method for reducing ferric salts prior to titration with the usual volumetric reagents. Occluded hydrogen has a similar though slower reducing power; and the nature of the metal by which the hydrogen is occluded is a factor in the reduction.

(4) *On the relation between the contraction of volume and the heat developed on mixing certain liquids.* By S. SKINNER, B.A., Christ's College.

The author, by calculation from known data, shewed that the contraction on mixing liquid hydrochloric acid with liquid water bore the same ratio (within 3 per cent.) to the heat which would be developed on mixing liquid hydrochloric acid and liquid water, in solutions varying in strength from 40 to ·4 per cent. of the acid.

He accounted for this by the hypothesis that some of the molecules of the acid form compound molecules with those of water. If this be true, the number of these compound molecules would be greater the less the acid present in proportion to the water. He shewed this on his hypothesis to be the case.

He also examined the cases of mixing liquid ammonia with liquid water, and liquid acetic acid with liquid water.

May 7, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

J. B. HOLT, B.A., Christ's College, was elected a Fellow of the Society.

The following communications were made:

(1) *On the Existence of Communications between the Body-cavity and the Vascular System.* By ARTHUR E. SHIPLEY, M.A.

In the General Considerations which follow Mr Sedgwick's recent paper upon the development of Peripatus Capensis, he sums up the characteristics of the coelom in the following terms: (i) the coelom does not communicate with the vascular system; (ii) it communicates with the exterior through nephridial pores; (iii) its lining gives rise to the generative products; (iv) it developes either as one or more diverticula from the primitive enteron, or as a space or spaces in the unsegmented or segmented mesoblastic bands (in the latter case called mesoblastic somites). Later on he calls attention to the fact that "there are certain animals to which the above general considerations as to the distinctness of the coelom and the vascular system do not apply." The animals here referred to are the Hirudinea and the Nemertea. In a later paper Sedgwick suggests the possibility that the nephridial funnels of Leeches might possibly open into a closed vesicle which lies in, but does not open into the vascular system. That some such structure may have been overlooked is rendered more probable when one recalls the number of able observers who failed to observe similar structures in Peripatus, and the fact that so careful a worker as Oscar Schultze overlooked the comparatively large nephridial funnels, when working at the excretory system of Clepsine.

Last term I devoted some time to the examination of these points. The forms I investigated were Clepsine, and to some extent Pontobdella amongst the Rhyncobdellidae, and amongst the Gnathobdellidae, Hirudo and Nephelis, and I may as well say at once that my researches on these forms confirm the results which Bourne published in the year 1884 in his exhaustive paper, "Contributions to the Anatomy of the Hirudinea[1]."

[1] *Quarterly Journal of Microscopical Science*, Vol. XXIV. p. 419.

The points to which I particularly directed my observation fall under three heads.

Firstly: Do the internal funnels really open, or end blindly, and in what spaces do their internal ends lie? For instance, are there any such sacs as Sedgwick has described enclosing the funnels of the nephridia of Peripatus?

Secondly: the communication between the true blood spaces and the sinuses, the nature of the fluid found in these spaces, and the circulation of the blood.

Thirdly: the embryological origin of the sinuses. With regard to this last I have been unable to make any investigation, but a certain amount of information on this subject is found in the writings of Nusbaum, Whitmann, and others.

With regard firstly to the nephridial funnels of Clepsine, I can fully confirm Bourne's statements. The funnel is usually composed of two cells, but in some cases I have seen three nuclei indicating the presence of three cells in the funnel; these surround a lumen; on one side this lumen is continuous with the sinus, and on the other hand with a sac. The lumen of the funnel is lined with long cilia. Bourne's figure of this structure is rather diagrammatic; the lumen of the funnel is occluded; but he definitely states that it opens, and in some of my preparations the coagulated mass of fluid in the sinus is joined to a similar coagulum in the sac mentioned above, by a strand of coagulated matter which in all respects resembles blood. The sac is usually full of coagulated fluid with small corpuscles scattered in it. In one nephridium there were two funnels, each opening into the sac; and again, I once saw a bunch of three or four funnels connected with the single sac of a nephridium.

The internal end of the nephridium of Hirudo does not open, but is surmounted by a number of cells, each with a depression. The fact that it does not open is regarded by Bourne as due to degeneration. This swollen end lies in a space which contains red blood, and there is no sac full of coagulated blood and corpuscles as in Clepsine.

Nephelis, however, is provided with nephridial funnels which do open on the one hand into the space in which their internal ends are situated, and on the other into a sac similar to that found in Clepsine, which contains both coagulum and corpuscles.

With regard to the spaces in which the funnels lie, there seems to me to be no doubt that Bourne's description is correct. In Clepsine, the funnels lie in pairs, in the ventral sinus, with the ventral vessel and nerve cord between them. No trace of any special sac, such as is found in Peripatus, is present.

In Nephelis the funnels open into a special enlargement of the botryoidal tissue, but there is no reason to regard this as anything

more than a development of the coelomic spaces, as Bourne has done.

Again, in Hirudo, where the funnels do not open, the blind internal end lies in a perinephrostomial sinus, which again possesses no characteristics which would justify the assumption that it is fundamentally different from other coelomic spaces.

Before passing on to consider the means of communication of the vascular and coelomic spaces, I wish to insert a few remarks upon the sacs which are present on the nephridia, which have internal open funnels, and in which numerous corpuscles from the blood are found. These corpuscles seem to be degenerating, and in some cases they appear rather more granular than the normal corpuscles in the blood.

It has occurred to me that we have to do here with a phenomenon similar to that which Durham[1] has described in Asterias rubens. The amoeboid corpuscles, after devouring some substance which it is to the advantage of the organism to excrete instead of working their own way to the exterior, are taken up by the open funnel of the nephridium, and in the sac they disintegrate and are eventually thrown out from the body. In Asteroideae, where there are no nephridia, the corpuscles work their way out through the body-wall.

We owe our knowledge of the paths by means of which the fluid passes from the blood vessels into the coelom chiefly to Lankester and Bourne. Besides the direct communications which exist in the Rhyncobdellidae, there is the communication by means of the botryoidal tissue which is seen at its best in the Gnathobdellidae. A fragment of the brown tissue of a Leech shews at once the connection of the lumen of the botryoidal tissue with that of the thin walled vessels. And my sections through Clepsine and Hirudo shew in numerous places the large openings by means of which the botryoidal tissue is put into communication with the sinuses, sometimes a continuous coagulum being found, lying half in one and half in the other system of spaces.

The same kind of blood is found in both the true vessels and the sinuses, except that, as Bourne points out, certain large corpuscles which occur in the sinuses of Clepsine and Pontobdella are not found in the blood vessels, being, as he suggests, too large to pass through the communicating channels.

The contraction of the dorsal vessel in its sinus can be seen without difficulty, and I have often watched the ventral vessel contract, sending the blood from before backward, whilst the current in the sinus surrounding the vessel flowed in the reverse

1 H. E. Durham, "The Emigration of Amoeboid Corpuscles in the Starfish," *Proc. Roy. Soc.* Vol. 43, p. 327.

direction. The fluid and corpuscles in both vessels and sinuses being apparently identical.

The foregoing facts fully corroborate Bourne's statements that the nephridia open into the sinuses, which in their turn are in communication with the blood vessels, and which contain the same fluid as the vascular system. With regard to the embryological nature of these spaces we are largely indebted to the researches of Nusbaum[1]. He describes in Clepsine the mesoblastic bands dividing into 33 somites. Each of these somites acquires a cavity which gradually increases in size. The walls of this cavity on the upper side, towards the endoderm, become only one cell thick, they form the splanchnopleure. The opposite wall, the somatopleure, that next the ectoderm, is however several cells thick.

The anterior wall of each somite fuses with the posterior wall of the preceding somite, and thus septa, comparable to those of the higher worms, are formed, and persist for a short time in embryonic life. Soon, however, the somites fuse with one another, and their cavities become continuous. Then the walls of the two lateral cavities which are thus formed, and at first are only in the ventral face of the embryo, commence to grow round the endoderm. Part of the tissue forming the septa persists as the dorso-ventral muscles. The spaces on each side, growing dorsally and ventrally, fuse, and, by the arrangement of the dorso-ventral muscles two longitudinal septa are formed which divide the common space into a dorsal, ventral, and two lateral sinuses. These are the blood sinuses, which by the development of the connective tissue and muscles become relatively much smaller in the adult than in the embryo.

Nusbaum further describes and figures the development of the dorsal and ventral vessel, both of which apparently arise as a solid cord of cells, proliferated from the splanchnic layer of the mesoderm, in the middle dorsal and ventral line. They subsequently acquire a lumen, and, separating off, lie in their respective sinuses.

The same author, in describing the development of the nephridia, points out that in the young embryo they appear in every somite, even in those which form the posterior sucker, where they subsequently abort.

Thus with regard to the origin of the space and the opening into it of the nephridia, the sinuses of the Hirudinea are truly coelomic, the embryological researches of Nusbaum confirming in a most striking way the predictions of Bourne.

If we turn to the third characteristic of a coelom, that "its lining gives rise to the generative products," the evidence is not quite so satisfactory. The origin of the reproductive cells is

[1] *Archives Slaves de Biologie*, Vol. 1, pp. 320 and 539.

probably an example of "precocious segregation." The sexual cells arise from the mesoblasts—the segment cells of Whitmann—which, arising posteriorly, multiply and pass forward till a heap of them is formed laterally in each somite. One pair of these form the ovary and seven pairs become testes. According to Nusbaum the tunica of the generative glands is formed at the expense of the mesoderm. This doubtless buds off corpuscles, just as it does into the sinus, and thus forms the colourless corpuscles which Bourne found in the fluid surrounding the true ovary. Nusbaum traces the oviduct and vas deferens back to nephridia.

I have attempted so far to shew firstly that there is no doubt that the old statements with regard to the blood system of Leeches being in communication with the sinus system is true, and secondly that the sinus system is coelomic in nature. So that with regard to the group Hirudinea, the vascular system is undoubtedly in communication with the coelom.

Let us now turn to the Nemertines, the second group of animals mentioned by Sedgwick as forming an exception to the rule that the blood system is independent of the coelom.

The nephridial system of these animals is not so definite in its arrangement as amongst the Hirudinea. Oudemans[1] has examined it in a great number of forms, and I have to some extent been able to confirm his observations. In his summary at the end of his paper he states, "the nephridial system of the Nemertea consists of one or more canals, directly communicating, or not, with the vascular system, provided, or not, with cilia, and communicating with the exterior by means of excretory ducts."

But when we come to consider the nature of these spaces which contain blood, and in which the internal end of the nephridium is sometimes situated, we shall see that they differ considerably in their fundamental origin from the sinus system of the Hirudinea.

In his valuable work on the embryology of Lineus obscurus, Hubrecht[2] points out that the blood vascular system together with the proboscidian cavity represents the last remnants of the archicoel or segmentation cavity. Hubrecht has proposed the name archi-coelom for this system of spaces, and in which, as is stated above, the inner ends of the nephridia sometimes lie.

I have already drawn attention[3] in a previous paper to the fact that the cavity of the heart in the embryo Lamprey is

[1] A. C. Oudemans, "The Circulatory and Nephridial Apparatus of the Nemertea." *Q. J. M. S.* 1885. Supplement.

[2] A. A. W. Hubrecht, "Contributions to the Embryology of Nemertea." *Quarterly Journal of Microscopical Science*, Vol. XXVI. p. 417.

[3] "On some points in the Development of Petromyzon Fluviatilis." *Q. J. M. S.* Vol. XXVII. p. 325.

continuous with the segmentation cavity. In my account of the development of the heart the following passage occurs: "From the fact...that the mesoblast behind the heart has not split into somatic and splanchnic layers, and not united ventrally, it will be seen that the cavity of the heart communicates posteriorly with the space between the ventral yolk cells (hypoblast) and the epidermis. Such a space would be equivalent to the segmentation cavity." Such a space exists, and becomes for a time crowded with blood corpuscles budded off from the free edges of the mesoblast, which occupies its dorso-lateral angles. These subsequently become enclosed in a secondary cavity formed by the down-growth and fusion of the mesoblastic laminae, and so come to lie in the heart and subintestinal veins.

When I wrote the above I was not aware that Bütschli[1] had conjectured that the cavity of the vascular system of Vertebrates was derived from the segmentation cavity. What he conceived from theoretical grounds I was able to see in the developing Lamprey. I think we are therefore justified in applying to the vascular system of Vertebrates the term archi-coelom, which Hubrecht has suggested for the blood-containing spaces in the Nemertea.

The system of spaces then of Nemertea which contain blood, and in which the inner ends of the nephridia sometimes lie, are not coelomic in their nature, but archicoelomic; and as the cavity-sheath of the proboscis has a similar origin, we are driven to the conclusion that there is no coelom in these animals, and therefore there can be no communication between the coelom and the vascular systems in this group, such as has been demonstrated for the Hirudinea.

The Gephyrea form another group of animals in which, like the Hirudinea, there is direct communication between the coelom and the blood vessels.

The body-cavity of Sipunculus is developed as a split in the mesoblastic bands; the cells lining it give rise to the generative products; and the nephridia open at their internal ends into it.

The blood vascular system arises late. Hatschek[2] describes its first origin during the metamorphosis of the larva, lying on the dorsal side of the alimentary canal. Although his description is not very detailed, there is nothing to shew that we have here to do with anything more than a normal blood vessel.

In the adult the main longitudinal vessel lies well surrounded

[1] "Ueber eine Hypothese bezüglich der phylogenetischen Herleitung des Blutgefässapparates eines Theils der Metazoen." *Morph. Jahrbuch*, Vol. 8, 1883.

[2] B. Hatschek, "Ueber Entwicklung von Sipunculus nudus." *Arbeiten aus dem Zoologischen Institut*, Wien, Bd. v. p. 33.

by connective tissue, and between two of the longitudinal vessels; it contains usually only blood corpuscles, which are exactly like those found freely in the body-cavity; but in individuals which are sexually ripe, spermatozoa and ova are often found in it. The openings, by means of which the cavity of this vessel communicates with the coelom, can be seen if the vessel be dissected out and exposed under a microscope; and further, Vogt and Yung[1] state that it is easy to inject the former from the latter.

Another group which stands far apart from both the Hirudinea and the Gephyrea, and in which communications exist between the vascular system and the cœlom, or at any rate with part of it, is the Echinodermata. Here, according to the observations of Hamann and Koehler, in Spatangids at least the blood system is in communication with the water vascular system, embryologically a part of the coelom and developed from an outgrowth of the body-cavity. And according to the French school of naturalists who have worked at this group, and amongst whom Perrier is the most prominent, this connection may be extended to the whole group of the Echinodermata.

Finally, in the class Vertebrata we again find the body-cavity, which is admittedly coelomic in nature, in communication with the vascular system, which is to some extent at any rate archicoelomic. The means of communication is through the lymphatic system. This opens on the one hand into the body-cavity by means of open stomata, and on the other by means of the thoracic duct into the venous system.

That fluids can pass from the body-cavity into the blood system by means of the lymphatic system has been shewn both by Recklingshausen and by Ludwig. The former found that milk put upon the peritoneal surface of the central tendon of the diaphragm—where numerous stomata exist—shewed little eddies caused by the milk globules passing through the stomata and entering the lymphatics. Ludwig's experiment is even more conclusive. He took a dead rabbit, and removed its viscera, and placed it so that the peritoneal surface of the diaphragm was exposed. He then poured into this a solution of Prussian blue, and, after imitating the respiratory movements for a few minutes, he obtained the lymphatics filled with a blue injection, shewing a beautiful plexus.

A more direct communication between the blood system and part of the body-cavity has been described in one Vertebrate. Weldon[2] has described and figured the structure of the head

[1] Vogt and Yung, "Lehrbuch der Praktischen Vergleichenden Anatomie."

[2] "On the Head Kidney of Bdellostoma," by W. F. R. Weldon. *Q. J. M. S.* Vol. 24, 1884.

kidney in Bdellostoma Forsteri. He finds running through the substance of this organ a number of fine tubules, lined with columnar cells and anastomosing with one another. These tubules open on the one hand into the pericardium and on the other into a central duct. In this duct lies a clot which is exactly similar to the blood clots found in the surrounding blood vessels. Further, in some cases capillaries were seen to enter this duct. There seems to be no reason to doubt that in this animal we have a part of the vascular system in communication with a part of the body-cavity through the tubules of the head kidney.

That there is a very primitive connection between these systems, is further supported by the remarkable observations of Seeliger[1], and Van Beneden and Julin[2] in the development of the heart of Clavellina.

These authors describe and figure in all stages the development of the heart and the pericardium of this Ascidian from an outgrowth of the ventral wall of that part of the endoderm which forms the pharynx, close to the end of the endostyle. This hollow diverticulum becomes separated from the endoderm and lies as a closed vesicle outside it. One half of the vesicle then invaginates, so that a two-walled vesicle results, there being a space left between the outer and inner wall. This space becomes the cavity of the pericardium, whose wall is formed of the outer layer of the double vesicle; this cavity is derived from the cavity of the endoderm.

The inner wall of the vesicle forms the wall of the heart, and the cavity of the heart is continuous with the primitive body-cavity. The longitudinal opening from the heart into the body-cavity persists for some time, until the free swimming larval stage; eventually it closes in the middle but still leaves an anterior and posterior opening through which the blood enters the heart from the body-cavity and leaves it again each time that organ contracts.

In Kleinenberg's[3] remarkable paper on the larva of Lopadorhyncus, he states that the segmentation cavity becomes the coelom in this and in many other Annelids. The coelom is therefore in these animals archi-coelic in nature, and we have seen that in some vertebrates the vascular system is of this nature. In the Nemertea the spaces which may be perhaps considered to be both body-cavity and vascular cavity are also archi-coelic. This group of animals would therefore seem to have retained the most primitive

[1] "Die Entwicklungsgeschichte der Socialen Ascidien," Oswald Seeliger. *Jenaische Zeitsch. für Naturwissenschaft*, 1885.

[2] "Recherches sur la Morphologie des Tuniciers," Van Beneden and Julin. Gand, 1886.

[3] "Die Entstehung des Annelids aus der Larve von Lopadorhyncus." Kleinenberg, *Zeit. f. Wis. Zoologie*, Bd. 44, 1886.

of all cavities—the segmentation cavity—as the only system of spaces between the endoderm and ectoderm: whilst the primitive segmentation cavity has differentiated in the higher animals, on the one hand into body-cavity—Annelids, and on the other in the cavities of the vascular system—Vertebrates.

(2) *On the Secretion of Salts in Saliva.* By J. N. LANGLEY, M.A., Trinity College, and H. M. FLETCHER, B.A., Trinity College.

May 21, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following communications were made:

(1) *On Solution and Crystallization.* By Professor LIVEING.

[*Abstract.*]

When a substance passes from a state of solution into the solid state, the new arrangement of the matter must be such that the entropy of the system is a maximum; and, other things being the same, the surface energy of the newly formed solid must be a minimum. If the surface tension be positive, that is tend to contract the surface, the surface energy will be a minimum when the approximation of the molecules of the surface is a maximum.

The essential difference between a solid and a fluid is that the molecules of the former maintain approximately the same relative places whereas the molecules of a fluid are subject to diffusion. Further, crystalloids in assuming the solid form assume a regular arrangement of their molecules throughout their mass, which we can usually recognise by the optical properties of the crystal, and by the cleavage. If we suppose space to be divided into cubes by three sets of parallel planes, each set at right angles to the other two, and suppose a molecule to be placed in every point where three planes intersect, we shall have an arrangement which corresponds with the isotropic character of a crystal of the cubic system. But of all the surfaces which can be drawn through the system the planes bounding the cubes meet the greatest number of molecules, those parallel to the faces of the dodecahedron meet the next greatest number of molecules, and those parallel to the faces of the octahedron meet the next greatest number. Also if we take an angular point of one of the cubes as origin, and the other three edges of the cube as axes, and the length of an edge of the cube as the unit of length, every plane

which cuts the three axes distances p, q, r respectively from the origin, where p, q and r are whole numbers, will be a surface of maximum concentration of molecules, but the concentration will be less as p, q and r are greater. Hence forms which are bounded by these planes, which follow the law of indices of crystals, will be forms of minimum surface energy and therefore of equilibrium. The tendency in general will be for substances with such a structure as is here supposed to take the form of cubes, since the cube will have the greatest concentration of molecules per unit of surface. But the total surface energy will depend on the total surface as well as on the energy per unit of surface, and for a given volume the surface will be diminished if the edges and angles of the cube are truncated by faces of the dodecahedron and octahedron, or by more complicated forms.

When a solid is broken two new surfaces are formed each with its own surface energy, and the solid must be more easily fractured when the new surfaces have the minimum energy. Hence substances with the structure supposed must break most easily in directions parallel to the sides of the cube, dodecahedron and octahedron: and these are the cleavages observed in this system. If we suppose the molecules placed at the centres of the faces of the cubes, instead of at the angles, the arrangement will still be isotropic, but the octahedron will be bounded by the surfaces of greatest condensation and the cube will come next to it. It is probable that substances which cleave most readily into cubes, such as rock salt and galena, have the former structure, while those which have the octahedral cleavage may have the latter arrangement of their molecules.

For the pyramidal and prismatic systems we may suppose space divided not into cubes but into rectangular parallelopipeds with edges equal severally to the axes of the crystals, and molecules placed as before. For the rhombohedral system we may suppose space divided into rhombohedra, or in crystals of the hexagonal type into right prisms with triangular bases, and for the other systems into parallelopipeds with edges parallel and equal to the axes. In each case if the molecules be disposed at points of intersection of three dividing planes we shall have such an arrangement as satisfies the optical conditions, and planes which follow the law of indices are surfaces of maximum condensation. Calculations show that whenever a crystal has an easily obtained cleavage the direction of cleavage corresponds to the surface of greatest condensation, and that the most common forms of crystals correspond in general to forms of minimum surface energy.

The surface tension of a plane surface will have no resultant out of that plane, but where two plane surfaces meet in an edge,

or angle, the tensions will have a resultant of sensible magnitude in some direction falling within the angle. Whenever all the faces of a crystallographic form are developed every such resultant will be met by an equal and opposite resultant and the form will be one of equilibrium. If one edge, or angle, be modified, the opposite edge, or angle, must either be similarly modified, or the resultant arising from the modification must be equilibrated by some internal forces produced by displacement of the molecules. In general equilibrium is attained by similar modifications of similar edges and angles, but when only some of the edges or angles of a crystal are modified while other similar edges or angles are not modified we usually have evidence of the consequent internal strain. Thus cubes of sodium chlorate which have half the angles truncated by faces of a tetrahedron rotate the plane of polarised light, hemihedral tourmalines are pyro-electric and so on. This theory therefore accounts for the plane faces of crystals, the law of indices, the most common combinations, and the cleavages. The same theory accounts for the development of plane faces when a crystalline solid of any shape is slowly acted on by a solvent. Solution will proceed so long as the entropy of the system is increased by the change, but when the solution is nearly saturated there will be an increase of entropy from the solution of a surface which has more than the minimum surface energy while there will be no increase from the solution of a surface which has only the minimum energy.

(2) *On the effect of an electric current on saturated solutions.* By C. Chree, M.A., King's College.

The following experiments were undertaken at the suggestion of Prof. J. J. Thomson, to whom I am much indebted for suggestions as to the salts employed and the method of carrying on the operations, which were conducted in the Cavendish Laboratory.

It has been shown by Prof. Thomson, from considerations based on a generalized form of Lagrange's equations, that a species of reciprocity may be expected in the interaction of natural agencies. Thus as the electrical resistance of a solution depends on the quantity of salt present, it seems desirable to know what effect the presence of an electric current has on the quantity of salt in solution.

It is obvious that the current will have an indirect effect owing to its heating the solution. The application of heat in general accelerates the rate of solution in an unsaturated solution, and by raising the point of saturation enables more salt to dissolve

in an originally saturated solution. Further the heating produced by an electric current is naturally more or less unequally distributed, and so must give rise to convection currents whose mechanical effect would naturally increase the rate of solution in an unsaturated solution.

In a saturated solution the indirect influences of the current seem likely to be a minimum, and accordingly in the following experiments none but saturated—or very nearly saturated—solutions were employed.

The salts examined were three in number, all chlorides, viz. of Sodium, Potassium and Calcium. The data as to their degrees of solubility and saturation are derived from Storer's *Dictionary of Solubilities*, and the data as to their electrical conductivities from Wiedemann's *Elektricität*, Erster Band. They were selected as representatives of various classes of salts.

In sodium chloride the quantity of salt required for saturation increases with the temperature but at a very slow rate. According to Gay Lussac the percentage by weight of salt in the saturated solution at 0° is 35·52 and at 50° is only 36·98. In potassium chloride according to the same observer the percentage of salt in a saturated solution rises from 29·21 at 0° to 43·59 at 52°·39. As to the quantity of salt in a saturated solution of calcium chloride different observers seem to vary considerably, but they all agree that it increases rapidly with the temperature.

In Wiedemann's *Elektricität* are curves whose abscissae give the amounts of salt in the solution and whose ordinates give the corresponding conductivities. Most, if not all, seem to agree in representing the conductivity as at first increasing with the amounts of salt but at a gradually diminishing rate. In some, e.g. calcium chloride, the conductivity attains a maximum and then diminishes before the point of saturation is reached. In others, e.g. potassium chloride, the form of the curve suggests that a maximum of conductivity would supervene but for its appearance being forestalled by the point of saturation being reached. In sodium chloride the conductivity is still increasing, but extremely slowly, as the point of saturation is approached. In calcium and in potassium chloride, especially the latter, the curve of conductivity is fairly steep near the point of saturation.

The object of the experiments being to detect what was soon seen to be a very small effect, strong currents were used so as to get as large an effect as possible. To prevent electrolysis a commutator giving a rapidly reversed current was employed. With comparatively weak currents there was in general no trace of electrolysis; but when, as sometimes happened, the current from 6 or more large storage cells passed between electrodes of several square inches in surface, at a distance of two inches apart in the

fluid, it was scarcely possible however fast the commutator was run to prevent some trace of bubbles forming on one or both of the electrodes. In no case of course was the electrolysis allowed to take place to any extent, but there seemed no sensible difference between results obtained on occasions on which it could not be seen and other occasions on which it could.

The electrodes were of platinized platinum foil and were several times subsequently replatinized, and the effect of this in getting rid of electrolysis was very marked.

The solution was contained in a glass jar, which was usually placed in a vessel containing water so as to check the heating effect of the current. The electrodes were supported so as nearly to reach the bottom of the glass jar, which was convex upwards. Fragments of the salt were always placed in the jar so as to lie between the electrodes, and an additional supply of the salt was suspended near the fluid surface on a flat wooden spoon in a position towards one side of the jar but still right in the path of the current. Samples of the fluid were taken out by means of a 10 c.c. pipette, and in removing them care was taken to hold the end of the pipette in the path of the current midway between the electrodes, and to keep it well clear of the wooden spoon and also of the bottom of the jar. The samples being taken at a distance of an inch or so from either electrode, the effects of any electrolysis were presumably in great measure done away with.

The general order of carrying out an experiment was as follows. A cover, used to keep out dust, was taken off the glass jar, the electrodes inserted, and the solution well stirred. It was then seen that a fair number of undissolved crystals, always kept in the solution, were between the electrodes, and the spoon with its load of salt was put in its place. After a lapse of several hours a sample of the solution was taken and the current was then started. After an interval of from 40 to 120 minutes a second sample was taken and the current was then stopped. Finally after another interval of from 20 to 60 minutes a third sample was taken. On taking each sample the time and the temperature of the solution were noted.

Great care was exercised in filling the pipette, and it was always emptied in the same way and washed out a certain number of times into a small flask which was then corked and labelled. The flask was subsequently washed out repeatedly into a half litre flask which was then filled up to the mark. From the half litre flask 10 or 20 c.c. were removed by the pipette, which was washed out into a small beaker and the contents titrated with argentic nitrate. As a rule distilled water was used in all the operations, but in some of the experiments on calcium chloride a large vessel was filled with undistilled water and its contents used

in all the operations dealing with the samples taken on one occasion. As the object aimed at was merely to detect differences between these samples, and the same plan was followed in treating each, this could introduce no error.

The argentic nitrate solution usually contained about 10 grammes of salt in 500 c.c. of water. This supplied enough for a large number of experiments.

No great care was exercised in keeping the solution at an exact strength, and though tightly corked it seemed to change slowly. During the treatment however of the samples taken on the same occasion no sensible change could occur, and as the titrations were always repeated and the samples taken in varying order even this small effect must have been pretty well eliminated. It was not attempted to determine the absolute amount of salt in solution but only to detect differences between the samples.

In carrying out the titration a drop or two of a solution of yellow chromate of potassa in distilled water was used as an indicator. The burette from which the argentic nitrate was run was graduated to $\frac{1}{10}$ c.c., and could be read with fair accuracy to $\frac{1}{4}$ of a graduation. In treating the sodium and potassium chlorides the argentic nitrate was kept very nearly at a constant strength, and the average of the results obtained is included in the following table:

	Before current		Time current run, in minutes	During current		Interval elapsed, in minutes	After current	
	Temp.	Vol. $AgNO_3$		Temp.	Vol. $AgNO_3$		Temp.	Vol. $AgNO_3$
NaCl	$19\frac{2}{3}°$	18·99	115	$23\frac{2}{3}°$	19·02	60	$22\frac{1}{2}°$	19·08
K Cl	$13\frac{3}{4}°$	13·47	46	$16\frac{1}{2}°$	13·49	28	$15\frac{2}{3}°$	13·55

By "Vol. $AgNO_3$" is meant the number of c.c. of argentic nitrate required to titrate 20 c.c. taken from the half litre to which the 10 c.c. taken from the saturated solution had been diluted. "Time current run" is the interval between starting the current and taking the sample during its passage. "Interval elapsed" is the interval between stopping the current and taking the last sample. The other headings will explain themselves. In every case it is the average of a number of experiments that is given.

It is obvious from the table that the effect of the current must be small. The accuracy of the method would scarcely justify us in accepting as proved any increase in the quantity of salt in solution during the passage of the current. On the other hand the increase after the current stopped seems unquestionable, though accompanied by a slight fall in temperature. This increase shewed itself in almost every experiment on the KCl.

According to Storer's data an increase of 1° at the temperature of the experiments should have increased the $AgNO_3$ equivalent of

the solutions, if always saturated, by ·013 c.c. in the case of NaCl and ·11 c.c. in the case of KCl.

As regards the NaCl it will be observed that the small apparent increase during the current in the quantity of salt in solution would be more than accounted for by the rise of temperature, but that the increase after the current stopped over the quantity in solution before the current began is greater than Storer's data would allow. This may have been due in part to slight supersaturation, but is probably mainly due to experimental errors. The only safe conclusions would seem to be that the effect of the electric current must be extremely small, and that the application of heat slightly increases the quantity of salt required for saturation. Also the heat produces its effect very slowly or else its action is checked by the current.

From the KCl experiments more definite conclusions can be deduced. Supposing the solution to have been always saturated the average sample taken during the current should have required at least ·3 c.c. and that after the current at least ·2 c.c. more $AgNO_3$ than that taken before the current passed. In the former case the actually observed increase was only ·02 c.c.—really rather less—and in the latter case ·08 c.c. That there was a certain small increase in the latter case can hardly be questioned, but it was decidedly less than should have occurred if the solution had been fully saturated at the temperature when the third sample was taken. Considering the slightly higher temperature when the current was stopped this indicates that the rise of temperature would take a long time to produce its full effect. This, as will be shown, was confirmed by subsequent experiment. The increase of ·02 c.c. during the current is too small to rely on with certainty. Considering that the rise of temperature was naturally fastest at first, and that the time the current ran was so much longer than the interval between the time it stopped and the taking of the third sample, we are led to hold, as at least highly probable, the view that the current retarded the solution of the salt.

In order to judge of the effect of heat alone, the same KCl solution was on a subsequent occasion placed in a vessel containing water. This was heated so that the temperature in the solution rose gradually from 10° to $15\frac{1}{3}$° in 45 minutes. The heating being stopped the temperature fell to 15° in the course of 30 minutes. The volumes of a different $AgNO_3$ solution required for the titration of samples taken at these three temperatures were on an average 12·83, 12·96 and 13·03 c.c. respectively. The rise of temperature here is considerably greater than the average rise in the electrical experiments, but the quantity of salt dissolved during the heating is so considerable that if the heating action of the current had produced its natural effect the result could hardly

have escaped detection. This affords strong independent evidence in favour of the view stated above. It is noteworthy that in this last experiment the quantity of salt that dissolved from first to last was less than two-fifths of that required to saturate the solution at its final temperature, yet $1\frac{1}{4}$ hours elapsed between taking the first and last samples and during the last 40 minutes the temperature was nearly constant.

In the analysis of the calcium chloride, as already explained, undistilled water was sometimes used and in treating the samples taken on separate occasions different strengths of $AgNO_3$ solution were employed. Also on two occasions the current was allowed to produce its full heating effect. Thus it is only between samples taken on the same occasion that a strict comparison holds. In the following table are shown the average of the results obtained by analysis.

Date of experiment	Before current		Time current run, in minutes	During current		Interval elapsed, in minutes	After current	
	Temp.	Vol. $AgNO_3$		Temp.	Vol. $AgNO_3$		Temp.	Vol. $AgNO_3$
Dec. 9,	*	16·725	65	$15\frac{1}{2}°$	16·825	†	*	17·175
... 12,	$12\frac{1}{2}°$	*	35	$15\frac{3}{4}°$	18·80	20	$15\frac{1}{3}°$	18·86
... 14,	$13\frac{3}{4}°$	18·95	60	$24\frac{1}{4}°$	18·95	20	$21\frac{1}{2}°$	19·025
... 16,	14°	24·30	42	$20\frac{1}{2}°$	24·30	20	19°	24·45

The solution was made some 4 or 5 days before the first observation, and yet by direct analysis it was found that between Dec. 9th and 12th the quantity of salt dissolved had increased at least $\frac{1}{2}$ per cent. Samples taken on the 14th showed almost certainly a slight increase in the strength of the solution since the 12th. Samples however taken two months subsequently showed no increase in strength since the 16th.

On two occasions, about the date when the samples last mentioned were taken, the solution was heated up and three samples taken precisely as in the case of the KCl solution, distilled water being employed. On each occasion the temperature was raised $8\frac{1}{4}°$, the heating lasting in the first experiment for 55 minutes and in the second for 45. After taking the second samples cooling was allowed, on the first occasion for 45 minutes with a fall of $1\frac{1}{2}°$ in temperature, and on the second occasion for 35 minutes with a fall of $\frac{1}{4}°$. The $AgNO_3$ solutions employed on the two occasions were slightly different, but 17 c.c. was about the average quantity required for titration. The results obtained agreed very well, and indicated on an average that the second sample required ·12 c.c. more $AgNO_3$ than the first, and the third ·2 c.c. more than the second.

It will be seen that the heating effect on these occasions was about the mean of the heating effects in the two last electrical

* Not observed. † Probably about 30, certainly under 60.

experiments, and the amounts of salt dissolved during the cooling —considering the much shorter time in the electrical experiments and the more rapid fall in temperature—were not dissimilar. During the passage of the current however no alteration in the strength of the solution was detected, while in the case of pure heating, a very decided alteration took place. The conclusion can hardly be avoided that the current acted as a check on the process of solution. Considering however the great rise in temperature the very slow effect of the heating, even when no current was present, seems very remarkable.

That the current cannot wholly prevent the salt dissolving under all circumstances is obvious from the experiment on Dec. 9th, but the increase after the current stopped was still more decided. On this occasion the solution was certainly not saturated.

The results of the experiments on potassium and calcium chlorides thus agree in representing the action of the current as most probably retarding the rate at which salt dissolves in a nearly saturated solution. They also emphasise the extreme slowness with which the salt naturally dissolves in such a solution.

The following specific gravities of saturated solutions, all referring to temperatures very close to 15°, were taken with a common hydrometer. The solutions had all contained undissolved salt for not less than three months prior to the observations:

Ammonium chloride	1·076.
Potassium	1·169.
Sodium	1·206.
Calcium	1·394.

(3) *On compounds of arsenious oxide with sulphuric anhydride.* By R. H. Adie, B.A., Trinity College.

Arsenious oxide is readily soluble in sulphuric acid of various strengths when they are heated together. Excess of the oxide first crystallizes out, and the mother liquor on agitation deposits crystals of fairly constant composition for each strength of acid.

Crystals were obtained corresponding to

$$As_2O_3 . 4SO_3,$$
$$As_2O_3 . 2SO_3,$$
$$As_2O_3 . SO_3,$$
$$7As_2O_3 . SO_3.$$

(4) *Orthogonal Systems of Curves and of Surfaces.* By J. BRILL, M.A., St John's College.

1. Suppose that we have an orthogonal system of curves, and let ξ and η be the parameters of the constituent families of the system. Then, adopting the usual notation, we have

$$\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 = h_1^2,$$

$$\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 = h_2^2,$$

$$\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0.$$

From these equations we obtain

$$\frac{\dfrac{\partial \xi}{\partial x}}{\dfrac{\partial \eta}{\partial y}} = -\frac{\dfrac{\partial \xi}{\partial y}}{\dfrac{\partial \eta}{\partial x}} = \pm \frac{\left\{\left(\dfrac{\partial \xi}{\partial x}\right)^2 + \left(\dfrac{\partial \xi}{\partial y}\right)^2\right\}^{\frac{1}{2}}}{\left\{\left(\dfrac{\partial \eta}{\partial x}\right)^2 + \left(\dfrac{\partial \eta}{\partial y}\right)^2\right\}^{\frac{1}{2}}} = \pm \frac{h_1}{h_2}.$$

Thus we have, either

$$h_2 \frac{\partial \xi}{\partial x} = h_1 \frac{\partial \eta}{\partial y} \text{ and } h_2 \frac{\partial \xi}{\partial y} = -h_1 \frac{\partial \eta}{\partial x},$$

or

$$h_2 \frac{\partial \xi}{\partial x} = -h_1 \frac{\partial \eta}{\partial y} \text{ and } h_2 \frac{\partial \xi}{\partial y} = h_1 \frac{\partial \eta}{\partial x}.$$

These two forms are virtually identical. In developing the general theory we will make use of the first form. It will be necessary, however, in applying the theory to any particular case, to examine carefully which parameter must be chosen for ξ, and which for η.

Now we have

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy} = \frac{h_2\left\{\dfrac{\partial \xi}{\partial x}dx + \dfrac{\partial \xi}{\partial y}dy\right\} + ih_1\left\{\dfrac{\partial \eta}{\partial x}dx + \dfrac{\partial \eta}{\partial y}dy\right\}}{dx + idy}$$

$$= \frac{\left\{h_2 \dfrac{\partial \xi}{\partial x} + ih_1 \dfrac{\partial \eta}{\partial x}\right\}(dx + idy)}{dx + idy}$$

$$= h_2 \frac{\partial \xi}{\partial x} + ih_1 \frac{\partial \eta}{\partial x} = h_1 \frac{\partial \eta}{\partial y} - ih_2 \frac{\partial \xi}{\partial y}.$$

Thus it is evident that the value of the expression

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy}$$

is independent of the ratio $dy : dx$. The same will be true of the expression

$$(p+iq)\frac{h_2 d\xi + ih_1 d\eta}{dx+idy}.$$

Writing t for $p+iq$, we will seek to determine its form so that the expression $t(h_2 d\xi + ih_1 d\eta)$ may be a perfect differential. The condition for this is

$$\frac{\partial}{\partial\eta}(th_2) = i\frac{\partial}{\partial\xi}(th_1),$$

i.e.
$$h_2\frac{\partial t}{\partial\eta} - ih_1\frac{\partial t}{\partial\xi} + t\left\{\frac{\partial h_2}{\partial\eta} - i\frac{\partial h_1}{\partial\xi}\right\} = 0.$$

This equation will enable us to obtain a suitable form for t, and its solution will involve an arbitrary function. Further, if we write $t(h_2 d\xi + ih_1 d\eta) = dw$, we have

$$\frac{dw}{dz} = (p+iq)\left\{h_2\frac{\partial\xi}{\partial x} + ih_1\frac{\partial\eta}{\partial x}\right\} = (p+iq)\left\{h_1\frac{\partial\eta}{\partial y} - ih_2\frac{\partial\xi}{\partial y}\right\},$$

where $z = x+iy$. Also it is clear that w will be a function of z. Thus, if we write $w = \lambda + i\mu$, we have

$$\frac{dw}{dz} = \frac{\partial\lambda}{\partial x} + i\frac{\partial\mu}{\partial x} = \frac{\partial\mu}{\partial y} - i\frac{\partial\lambda}{\partial y}.$$

Comparing these two sets of equations, we obtain

$$ph_2\frac{\partial\xi}{\partial x} - qh_1\frac{\partial\eta}{\partial x} = ph_1\frac{\partial\eta}{\partial y} + qh_2\frac{\partial\xi}{\partial y} = \frac{\partial\lambda}{\partial x} = \frac{\partial\mu}{\partial y}$$

and
$$qh_2\frac{\partial\xi}{\partial x} + ph_1\frac{\partial\eta}{\partial x} = qh_1\frac{\partial\eta}{\partial y} - ph_2\frac{\partial\xi}{\partial y} = \frac{\partial\mu}{\partial x} = -\frac{\partial\lambda}{\partial y}.$$

These equations may also be written in the form

$$h_2\frac{\partial\xi}{\partial x} = \quad h_1\frac{\partial\eta}{\partial y} = m\frac{\partial\lambda}{\partial x} + n\frac{\partial\mu}{\partial x} = m\frac{\partial\mu}{\partial y} - n\frac{\partial\lambda}{\partial y}$$

and
$$h_2\frac{\partial\xi}{\partial y} = -h_1\frac{\partial\eta}{\partial x} = n\frac{\partial\lambda}{\partial x} - m\frac{\partial\mu}{\partial x} = n\frac{\partial\mu}{\partial y} + m\frac{\partial\lambda}{\partial y},$$

where $m = p/(p^2+q^2)$ and $n = q/(p^2+q^2)$. From these latter forms we deduce

$$h_1^2 h_2^2 = (m^2+n^2)\left\{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial x}\right)^2\right\} = \frac{1}{p^2+q^2}\left\{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right\}.$$

Thus if we write

$$h^2 = \left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 = \left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial y}\right)^2,$$

and $k^2 = p^2+q^2$, we have $h = h_1 h_2 k$.

2. If we separate the equation

$$h_2 \frac{\partial t}{\partial \eta} - ih_1 \frac{\partial t}{\partial \xi} + t\left\{\frac{\partial h_2}{\partial \eta} - i\frac{\partial h_1}{\partial \xi}\right\} = 0$$

into its real and imaginary parts, we obtain

$$h_2 \frac{\partial p}{\partial \eta} + h_1 \frac{\partial q}{\partial \xi} + p\frac{\partial h_2}{\partial \eta} + q\frac{\partial h_1}{\partial \xi} = 0$$

and

$$h_2 \frac{\partial q}{\partial \eta} - h_1 \frac{\partial p}{\partial \xi} + q\frac{\partial h_2}{\partial \eta} - p\frac{\partial h_1}{\partial \xi} = 0.$$

From these we easily deduce

$$h_2\left\{p\frac{\partial p}{\partial \eta} + q\frac{\partial q}{\partial \eta}\right\} + h_1\left\{p\frac{\partial q}{\partial \xi} - q\frac{\partial p}{\partial \xi}\right\} + (p^2+q^2)\frac{\partial h_2}{\partial \eta} = 0$$

and

$$h_2\left\{q\frac{\partial p}{\partial \eta} - p\frac{\partial q}{\partial \eta}\right\} + h_1\left\{p\frac{\partial p}{\partial \xi} + q\frac{\partial q}{\partial \xi}\right\} + (p^2+q^2)\frac{\partial h_1}{\partial \xi} = 0,$$

which may be written in the form

$$h_2\frac{\partial \log k}{\partial \eta} + h_1\frac{\partial}{\partial \xi}\tan^{-1}\frac{q}{p} + \frac{\partial h_2}{\partial \eta} = 0$$

and

$$h_2\frac{\partial}{\partial \eta}\tan^{-1}\frac{q}{p} - h_1\frac{\partial \log k}{\partial \xi} - \frac{\partial h_1}{\partial \xi} = 0.$$

Thus we have the two equations

$$\frac{\partial}{\partial \xi}\left\{\frac{h_1}{h_2}\frac{\partial \log k}{\partial \xi} + \frac{1}{h_2}\frac{\partial h_1}{\partial \xi}\right\} + \frac{\partial}{\partial \eta}\left\{\frac{h_2}{h_1}\frac{\partial \log k}{\partial \eta} + \frac{1}{h_1}\frac{\partial h_2}{\partial \eta}\right\} = 0$$

and

$$\frac{\partial}{\partial \xi}\left\{\frac{h_1}{h_2}\frac{\partial}{\partial \xi}\tan^{-1}\frac{q}{p} + \frac{1}{h_2}\frac{\partial h_2}{\partial \eta}\right\} + \frac{\partial}{\partial \eta}\left\{\frac{h_2}{h_1}\frac{\partial}{\partial \eta}\tan^{-1}\frac{q}{p} - \frac{1}{h_1}\frac{\partial h_1}{\partial \xi}\right\} = 0.$$

3. From the equations of Art. 1, we deduce

$$h_1h_2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} = h_2^2\left(\frac{\partial \xi}{\partial x}\right)^2 \text{ and } -h_1h_2\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} = h_2^2\left(\frac{\partial \xi}{\partial y}\right)^2.$$

Thus we have

$$h_1h_2\frac{\partial(\xi, \eta)}{\partial(x, y)} = h_1^2h_2^2,$$

and therefore

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = h_1h_2.$$

We also have

$$\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}=\frac{\partial}{\partial x}\left\{\frac{h_1}{h_2}\,\frac{\partial \eta}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\frac{h_1}{h_2}\,\frac{\partial \eta}{\partial x}\right\}$$

$$=\frac{\partial}{\partial x}\left(\frac{h_1}{h_2}\right)\frac{\partial \eta}{\partial y}+\frac{h_1}{h_2}\,\frac{\partial^2 \eta}{\partial x \partial y}-\frac{\partial}{\partial y}\left(\frac{h_1}{h_2}\right)\frac{\partial \eta}{\partial x}-\frac{h_1}{h_2}\,\frac{\partial^2 \eta}{\partial x \partial y}$$

$$=\frac{\partial\,(h_1/h_2,\,\eta)}{\partial\,(x,\,y)}=h_1h_2\,\frac{\partial\,(h_1/h_2,\,\eta)}{\partial\,(\xi,\,\eta)}=h_1h_2\,\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2}\right).$$

Similarly we should obtain

$$\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}=h_1h_2\,\frac{\partial}{\partial \eta}\left(\frac{h_2}{h_1}\right).$$

Thus we have

$$\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}=\frac{\partial^2 Q}{\partial \xi^2}\left\{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\right\}+2\,\frac{\partial^2 Q}{\partial \xi \partial \eta}\left\{\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial x}+\frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial y}\right\}$$

$$+\frac{\partial^2 Q}{\partial \eta^2}\left\{\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2\right\}+\frac{\partial Q}{\partial \xi}\left\{\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}\right\}+\frac{\partial Q}{\partial \eta}\left\{\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}\right\}$$

$$=h_1^{\,2}\frac{\partial^2 Q}{\partial \xi^2}+h_2^{\,2}\frac{\partial^2 Q}{\partial \eta^2}+h_1h_2\left\{\frac{\partial Q}{\partial \xi}\,\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2}\right)+\frac{\partial Q}{\partial \eta}\,\frac{\partial}{\partial \eta}\left(\frac{h_2}{h_1}\right)\right\}.$$

4. If in the equation

$$\frac{\partial^2 \log h}{\partial x^2}+\frac{\partial^2 \log h}{\partial y^2}=0$$

we substitute for h the value found in Art. 1, and utilize the transformation of the preceding article, we have

$$\left\{h_1^{\,2}\frac{\partial^2}{\partial \xi^2}+h_2^{\,2}\frac{\partial^2}{\partial \eta^2}\right\}.\log h_1h_2k$$

$$+h_1h_2\left\{\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2}\right)\frac{\partial}{\partial \xi}(\log h_1h_2k)+\frac{\partial}{\partial \eta}\left(\frac{h_2}{h_1}\right)\frac{\partial}{\partial \eta}(\log h_1h_2k)\right\}=0.$$

By means of the first of the two equations at the end of Art. 2, we have

$$h_1^{\,2}\frac{\partial^2 \log k}{\partial \xi^2}+h_2^{\,2}\frac{\partial^2 \log k}{\partial \eta^2}+h_1h_2\left\{\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2}\right)\frac{\partial \log k}{\partial \xi}+\frac{\partial}{\partial \eta}\left(\frac{h_2}{h_1}\right)\frac{\partial \log k}{\partial \eta}\right\}$$

$$=-h_1h_2\left[\frac{\partial}{\partial \xi}\left\{\frac{1}{h_2}\,\frac{\partial h_1}{\partial \xi}\right\}+\frac{\partial}{\partial \eta}\left\{\frac{1}{h_1}\,\frac{\partial h_2}{\partial \eta}\right\}\right].$$

Thus our equation becomes

$$\left\{h_1^2\frac{\partial^2}{\partial\xi^2}+h_2^2\frac{\partial^2}{\partial\eta^2}\right\}.\log h_1h_2$$
$$+h_1h_2\left\{\frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2}\right)\frac{\partial}{\partial\xi}(\log h_1h_2)+\frac{\partial}{\partial\eta}\left(\frac{h_2}{h_1}\right)\frac{\partial}{\partial\eta}(\log h_1h_2)\right\}$$
$$=h_1h_2\left[\frac{\partial}{\partial\xi}\left\{\frac{1}{h_2}\,\frac{\partial h_1}{\partial\xi}\right\}+\frac{\partial}{\partial\eta}\left\{\frac{1}{h_1}\,\frac{\partial h_2}{\partial\eta}\right\}\right].$$

Expanding and reducing this equation, it becomes

$$\frac{h_1^2}{h_2}\,\frac{\partial^2h_2}{\partial\xi^2}+\frac{h_2^2}{h_1}\,\frac{\partial^2h_1}{\partial\eta^2}+\frac{h_1}{h_2}\,\frac{\partial h_1}{\partial\xi}\,\frac{\partial h_2}{\partial\xi}+\frac{h_2}{h_1}\,\frac{\partial h_1}{\partial\eta}\,\frac{\partial h_2}{\partial\eta}$$
$$-2\frac{h_1^2}{h_2^2}\left(\frac{\partial h_2}{\partial\xi}\right)^2-2\frac{h_2^2}{h_1^2}\left(\frac{\partial h_1}{\partial\eta}\right)^2=0.$$

This may be written in the form

$$h_1\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{h_2}\right)+h_2\frac{\partial^2}{\partial\eta^2}\left(\frac{1}{h_1}\right)+\frac{\partial h_1}{\partial\xi}\,\frac{\partial}{\partial\xi}\left(\frac{1}{h_2}\right)+\frac{\partial h_2}{\partial\eta}\,\frac{\partial}{\partial\eta}\left(\frac{1}{h_1}\right)=0,$$

which is Lamé's equation.

In a paper presented to the Society last term, I shewed that if we express the equation

$$\frac{\partial^2\log h}{\partial x^2}+\frac{\partial^2\log h}{\partial y^2}=0$$

in terms of the elements of the (λ, μ) system of curves, we obtain the modified form which Lamé's equation takes when applied to that system. I have here shewn that if we express the same equation in terms of the elements of the (ξ, η) system, we obtain the general form of Lamé's equation. This would suggest the possibility that some of the other equations obtained in my former paper had analogues in the general theory of orthogonal curves. The work connected with the reduction of those equations when transformed into our present system of co-ordinates would, however, be very tedious, and it would not be *a priori* certain that the portion containing k could be eliminated.

5. Suppose that we have a second orthogonal system of curves, α and β being the parameters of the two families of the system. Also, let

$$\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2=k_1^2$$

and

$$\left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2=k_2^2.$$

Then the values of the two expressions

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy} \text{ and } \frac{k_2 d\alpha + ik_1 d\beta}{dx + idy}$$

are each independent of the ratio $dy : dx$. Hence, if we suppose them both to refer to any, the same, variation made in an arbitrary direction from a given point, the value of the expression

$$\frac{h_2 d\xi + ih_1 d\eta}{k_2 d\alpha + ik_1 d\beta}$$

will be independent of the direction of that variation; *i.e.* the value of the said expression will be independent of the ratio $d\alpha : d\beta$. This gives us the two relations

$$h_2 k_1 \frac{\partial \xi}{\partial \alpha} = h_1 k_2 \frac{\partial \eta}{\partial \beta} \text{ and } h_2 k_2 \frac{\partial \xi}{\partial \beta} = -h_1 k_1 \frac{\partial \eta}{\partial \alpha}.$$

6. We will now attempt to develope with respect to orthogonal systems of curves traced upon any surface whatsoever, a theory analogous to that we have developed with reference to orthogonal systems of curves traced upon a plane. Let ξ and η be the parameters of the two families belonging to one orthogonal system, and α and β those of the two families belonging to another orthogonal system. The length of an elementary arc will be given by a formula of the type

$$ds^2 = \frac{d\xi^2}{h_1^{\ 2}} + \frac{d\eta^2}{h_2^{\ 2}}$$

in the first system of co-ordinates, and by a formula of the type

$$ds^2 = \frac{d\alpha^2}{k_1^{\ 2}} + \frac{d\beta^2}{k_2^{\ 2}}$$

in the second system of co-ordinates. Hence if we express any given elementary arc in terms of the elements of both systems and equate the results, we have the equation

$$\frac{1}{h_1^{\ 2}} \left\{ \frac{\partial \xi}{\partial \alpha} d\alpha + \frac{\partial \xi}{\partial \beta} d\beta \right\}^2 + \frac{1}{h_2^{\ 2}} \left\{ \frac{\partial \eta}{\partial \alpha} d\alpha + \frac{\partial \eta}{\partial \beta} d\beta \right\}^2 = \frac{d\alpha^2}{k_1^{\ 2}} + \frac{d\beta^2}{k_2^{\ 2}}.$$

This equation holds independently of the value of the ratio $d\alpha : d\beta$. Thus we have

$$\frac{1}{h_1^{\ 2}} \left(\frac{\partial \xi}{\partial \alpha} \right)^2 + \frac{1}{h_2^{\ 2}} \left(\frac{\partial \eta}{\partial \alpha} \right)^2 = \frac{1}{k_1^{\ 2}},$$

$$\frac{1}{h_1^{\ 2}} \left(\frac{\partial \xi}{\partial \beta} \right)^2 + \frac{1}{h_2^{\ 2}} \left(\frac{\partial \eta}{\partial \beta} \right)^2 = \frac{1}{k_2^{\ 2}},$$

$$\frac{1}{h_1^{\ 2}} \frac{\partial \xi}{\partial \alpha} \frac{\partial \xi}{\partial \beta} + \frac{1}{h_2^{\ 2}} \frac{\partial \eta}{\partial \alpha} \frac{\partial \eta}{\partial \beta} = 0.$$

From these equations we obtain

$$\frac{\dfrac{1}{h_1}\dfrac{\partial\xi}{\partial\alpha}}{\dfrac{1}{h_2}\dfrac{\partial\eta}{\partial\beta}} = -\frac{\dfrac{1}{h_2}\dfrac{\partial\eta}{\partial\alpha}}{\dfrac{1}{h_1}\dfrac{\partial\xi}{\partial\beta}} = \pm\frac{\left\{\dfrac{1}{h_1^2}\left(\dfrac{\partial\xi}{\partial\alpha}\right)^2+\dfrac{1}{h_2^2}\left(\dfrac{\partial\eta}{\partial\alpha}\right)^2\right\}^{\frac{1}{2}}}{\left\{\dfrac{1}{h_1^2}\left(\dfrac{\partial\xi}{\partial\beta}\right)^2+\dfrac{1}{h_2^2}\left(\dfrac{\partial\eta}{\partial\beta}\right)^2\right\}^{\frac{1}{2}}} = \pm\frac{\dfrac{1}{k_1}}{\dfrac{1}{k_2}}.$$

If we adopt the upper sign, we have the two relations

$$h_2k_1\frac{\partial\xi}{\partial\alpha} = h_1k_2\frac{\partial\eta}{\partial\beta} \text{ and } h_2k_2\frac{\partial\xi}{\partial\beta} = -h_1k_1\frac{\partial\eta}{\partial\alpha}.$$

These relations enable us to write the above equations in a different form, viz.

$$k_1^2\left(\frac{\partial\xi}{\partial\alpha}\right)^2 + k_2^2\left(\frac{\partial\xi}{\partial\beta}\right)^2 = h_1^2,$$

$$k_1^2\left(\frac{\partial\eta}{\partial\alpha}\right)^2 + k_2^2\left(\frac{\partial\eta}{\partial\beta}\right)^2 = h_2^2,$$

$$k_1^2\frac{\partial\xi}{\partial\alpha}\frac{\partial\eta}{\partial\alpha} + k_2^2\frac{\partial\xi}{\partial\beta}\frac{\partial\eta}{\partial\beta} = 0.$$

Now we have

$$\begin{aligned}
h_2d\xi + ih_1d\eta &= h_2\left\{\frac{\partial\xi}{\partial\alpha}d\alpha + \frac{\partial\xi}{\partial\beta}d\beta\right\} + ih_1\left\{\frac{\partial\eta}{\partial\alpha}d\alpha + \frac{\partial\eta}{\partial\beta}d\beta\right\} \\
&= h_2\frac{\partial\xi}{\partial\alpha}d\alpha - \frac{h_1k_1}{k_2}\frac{\partial\eta}{\partial\alpha}d\beta + ih_1\frac{\partial\eta}{\partial\alpha}d\alpha + i\frac{h_2k_1}{k_2}\frac{\partial\xi}{\partial\alpha}d\beta \\
&= \frac{h_2}{k_2}\frac{\partial\xi}{\partial\alpha}(k_2d\alpha + ik_1d\beta) + i\frac{h_1}{k_2}\frac{\partial\eta}{\partial\alpha}(k_2d\alpha + ik_1d\beta) \\
&= \frac{1}{k_2}\left\{h_2\frac{\partial\xi}{\partial\alpha} + ih_1\frac{\partial\eta}{\partial\alpha}\right\}(k_2d\alpha + ik_1d\beta).
\end{aligned}$$

Therefore, we have

$$\frac{h_2d\xi + ih_1d\eta}{k_2d\alpha + ik_1d\beta} = \frac{1}{k_2}\left\{h_2\frac{\partial\xi}{\partial\alpha} + ih_1\frac{\partial\eta}{\partial\alpha}\right\} = \frac{1}{k_1}\left\{h_1\frac{\partial\eta}{\partial\beta} - ih_2\frac{\partial\xi}{\partial\beta}\right\}.$$

7. The fact that the equations connected with our present theory are identical in form with those connected with the most general theory for the plane, as developed in Art. 5, would lead us to expect that the simpler forms of the plane theory also had their exact analogues in the theory relating to surfaces. That we may develope a theory exactly analogous to the theory of functions of a complex variable as applied to the plane, has been shewn by

Beltrami*. We will, however, shew how this theory may be deduced from our present results.

The value of the expression

$$\frac{(m+in)(h_2 d\xi + ih_1 d\eta)}{(p+iq)(k_2 d\alpha + ik_1 d\beta)}$$

will be independent of the value of the ratio $d\alpha : d\beta$; and, writing $f = m + in$ and $g = p + iq$, we will seek to determine f and g so that the expressions $f(h_2 d\xi + ih_1 d\eta)$ and $g(k_2 d\alpha + ik_1 d\beta)$ may be perfect differentials. The necessary conditions for this will be

$$\frac{\partial}{\partial\eta}(fh_2) = i\frac{\partial}{\partial\xi}(fh_1) \text{ and } \frac{\partial}{\partial\beta}(gk_2) = i\frac{\partial}{\partial\alpha}(gk_1),$$

or

$$h_2\frac{\partial f}{\partial\eta} - ih_1\frac{\partial f}{\partial\xi} + f\left\{\frac{\partial h_2}{\partial\eta} - i\frac{\partial h_1}{\partial\xi}\right\} = 0$$

and

$$k_2\frac{\partial g}{\partial\beta} - ik_1\frac{\partial g}{\partial\alpha} + g\left\{\frac{\partial k_2}{\partial\beta} - i\frac{\partial k_1}{\partial\alpha}\right\} = 0.$$

These equations will enable us to determine suitable forms for f and g, and the solution of each will involve an arbitrary function. Further, if we write

$$dw = f(h_2 d\xi + ih_1 d\eta) \text{ and } d\zeta = g(k_2 d\alpha + ik_1 d\beta),$$

then $dw/d\zeta$ will possess a single definite value; and, if we further write $w = \lambda + i\mu$ and $\zeta = \gamma + i\delta$, we see that the value of the expression

$$\frac{d\lambda + id\mu}{d\gamma + id\delta}$$

is independent of the value of the ratio $d\gamma : d\delta$. This necessitates the relations

$$\frac{\partial\lambda}{\partial\gamma} = \frac{\partial\mu}{\partial\delta} \text{ and } \frac{\partial\lambda}{\partial\delta} = -\frac{\partial\mu}{\partial\gamma}.$$

Thus we see that $\lambda + i\mu = F(\gamma + i\delta)$; and, consequently, we are furnished with a theory exactly analogous to the theory of functions of a complex variable as applied to the plane.

Separating the equation

$$(p+iq)(k_2 d\alpha + ik_1 d\beta) = d\gamma + id\delta$$

into its real and imaginary parts, we have

$$pk_2 d\alpha - qk_1 d\beta = d\gamma \text{ and } qk_2 d\alpha + pk_1 d\beta = d\delta.$$

* "Delle variabili complesse sopra una superficie qualunque"—*Annali di Matematica* (2) I., 329—366. The essential elements of the theory are, however, due to Gauss.

Therefore

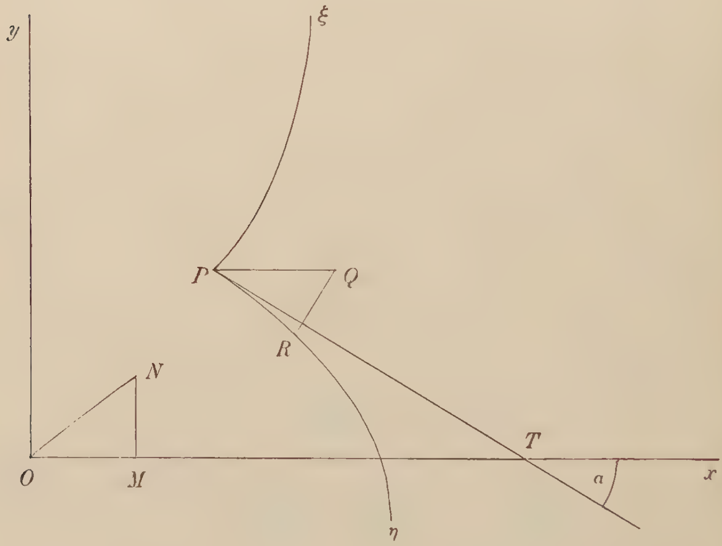
$$d\alpha = \frac{pd\gamma + qd\delta}{(p^2+q^2)k_2} \text{ and } d\beta = \frac{pd\delta - qd\gamma}{(p^2+q^2)k_1}.$$

Hence, if ds be any elementary arc, we have

$$ds^2 = \frac{d\alpha^2}{k_1^2} + \frac{d\beta^2}{k_2^2} = \frac{(pd\gamma + qd\delta)^2 + (pd\delta - qd\gamma)^2}{(p^2+q^2)^2 k_1^2 k_2^2} = \frac{d\gamma^2 + d\delta^2}{(p^2+q^2)k_1^2 k_2^2}.$$

Similarly we should have $ds^2 = \dfrac{d\lambda^2 + d\mu^2}{(m^2+n^2)h_1^2 h_2^2}$.

8. In order to discover the manner in which the theory that we have developed for the plane may be imitated in geometry of three dimensions, we will seek to determine the geometrical meaning of that theory.



Let P be any point, and through it draw a straight line PQ of infinitesimal length, in any direction. Through P draw a curve belonging to the family whose parameter is η, and through Q draw a curve belonging to the family whose parameter is ξ. Let these two curves meet in R; and let $PR = ds_1$ and $QR = ds_2$. Take a length OM on the axis of x equal to PR, and through M draw MN parallel to the axis of y and equal in length to QR. Join ON. Then the elementary triangles PQR and OMN are

equal in all respects. Thus ON and PQ are of equal length, and make with each other an angle α equal to that made by the tangent at P to PR with the axis of x (see above figure). Thus

$$\frac{(ON)}{(PQ)} = \cos\alpha + i\sin\alpha;$$

i.e. the ratio of the vectors (ON) and (PQ) is independent of the direction in which PQ is drawn. And we have

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy} = h_1 h_2 \,.\, \frac{ds_1 + ids_2}{dx + idy} = h_1 h_2 \,.\, \frac{(ON)}{(PQ)}\,.$$

Thus we see that the fact that the expression

$$\frac{h_2 d\xi + ih_1 d\eta}{dx + idy}$$

possesses a definite value which is independent of the ratio $dy : dx$, depends upon the fact that the vector (PQ) may be brought into parallelism with the vector (ON) by a twist whose magnitude depends only upon the position of the point P, and not upon the direction of PQ.

Now suppose that we have an orthogonal system of surfaces. Draw a pair of consecutive surfaces belonging to each family. These six surfaces will enclose an elementary rectangular parallelopiped. Let P and Q be two opposite vertices of this parallelopiped, and let PR, RS, SQ be consecutive edges of it. Draw another parallelopiped having its edges OL, LM, MN parallel to the axes of co-ordinates, and respectively equal to PR, RS, SQ. The two parallelopipeds will be equal in all respects, and ON will be equal in length to PQ. Now the directions of PR, RS, SQ depend only on the position of the point P, and not upon the direction of the line PQ. Thus, whatever the direction of the line PQ, a single definite twist about some determinate axis will bring the edges of one parallelopiped into parallelism with the corresponding edges of the other, and therefore PQ into parallelism with ON. This twist will, however, no longer be represented by the ratio of the vectors $\overline{ON}$ and $\overline{PQ}$, since that ratio would be a quaternion having its axis in the direction of the normal to a plane drawn parallel to ON and PQ, and would therefore be a variable quantity. On the contrary, if we write $\overline{PQ} = d\rho$ and $\overline{ON} = d\sigma$, we shall have an equation of the form $d\sigma = q\,.\,d\rho\,.\,q^{-1}$, or $q\,.\,d\rho = d\sigma\,.\,q$, where q is a quaternion whose value depends only upon the position of the point P, and not upon the direction of PQ.

9. Let ξ, η, ζ be the parameters of the three families of surfaces forming our orthogonal system. We have the equations

$$\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2=h_1^2,$$
$$\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2=h_2^2,$$
$$\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2+\left(\frac{\partial\zeta}{\partial z}\right)^2=h_3^2.$$
$$\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}+\frac{\partial\eta}{\partial z}\frac{\partial\zeta}{\partial z}=0,$$
$$\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial x}+\frac{\partial\zeta}{\partial y}\frac{\partial\xi}{\partial y}+\frac{\partial\zeta}{\partial z}\frac{\partial\xi}{\partial z}=0,$$
$$\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial z}=0;$$

which may also be written in the form

$$\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial x}\right)^2=1,$$
$$\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial y}\right)^2+\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial y}\right)^2+\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial y}\right)^2=1,$$
$$\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial z}\right)^2+\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial z}\right)^2+\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial z}\right)^2=1,$$
$$\frac{1}{h_1^2}\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}+\frac{1}{h_2^2}\frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial z}+\frac{1}{h_3^2}\frac{\partial\zeta}{\partial y}\frac{\partial\zeta}{\partial z}=0,$$
$$\frac{1}{h_1^2}\frac{\partial\xi}{\partial z}\frac{\partial\xi}{\partial x}+\frac{1}{h_2^2}\frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial x}+\frac{1}{h_3^2}\frac{\partial\zeta}{\partial z}\frac{\partial\zeta}{\partial x}=0,$$
$$\frac{1}{h_1^2}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+\frac{1}{h_2^2}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+\frac{1}{h_3^2}\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}=0.$$

From the last three equations of the first group we obtain

$$\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial(\eta,\zeta)}{\partial(y,z)}}=\frac{\frac{\partial\xi}{\partial y}}{\frac{\partial(\eta,\zeta)}{\partial(z,x)}}=\frac{\frac{\partial\xi}{\partial z}}{\frac{\partial(\eta,\zeta)}{\partial(x,y)}}=\frac{1}{u},$$
$$\frac{\frac{\partial\eta}{\partial x}}{\frac{\partial(\zeta,\xi)}{\partial(y,z)}}=\frac{\frac{\partial\eta}{\partial y}}{\frac{\partial(\zeta,\xi)}{\partial(z,x)}}=\frac{\frac{\partial\eta}{\partial z}}{\frac{\partial(\zeta,\xi)}{\partial(x,y)}}=\frac{1}{v},$$

$$\frac{\frac{\partial\zeta}{\partial x}}{\frac{\partial(\xi,\eta)}{\partial(y,z)}}=\frac{\frac{\partial\zeta}{\partial y}}{\frac{\partial(\xi,\eta)}{\partial(z,x)}}=\frac{\frac{\partial\zeta}{\partial z}}{\frac{\partial(\xi,\eta)}{\partial(x,y)}}=\frac{1}{w}.$$

By combining the equations

$$\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial z}-\frac{\partial\eta}{\partial z}\frac{\partial\zeta}{\partial y}=u\frac{\partial\xi}{\partial x}$$

and

$$\frac{\partial\eta}{\partial z}\frac{\partial\zeta}{\partial x}-\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial z}=u\frac{\partial\xi}{\partial y},$$

we obtain

$$\frac{\partial\zeta}{\partial z}\left\{\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial x}\right\}-\frac{\partial\eta}{\partial z}\left\{\left(\frac{\partial\zeta}{\partial y}\right)^2+\left(\frac{\partial\zeta}{\partial x}\right)^2\right\}=u\left\{\frac{\partial\xi}{\partial x}\frac{\partial\zeta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\zeta}{\partial x}\right\},$$

i.e.

$$\frac{\partial\eta}{\partial z}\left\{\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2+\left(\frac{\partial\zeta}{\partial z}\right)^2\right\}=u\frac{\partial(\zeta,\xi)}{\partial(x,y)},$$

or

$$u\frac{\partial(\zeta,\xi)}{\partial(x,y)}=h_3^2\frac{\partial\eta}{\partial z}.$$

Thus we have $h_3^2=uv$; and similarly we should obtain $h_1^2=vw$ and $h_2^2=wu$. Combining these we have $h_1^2h_2^2=h_3^2w^2$; and, consequently, we obtain the equations

$$u=\frac{h_2h_3}{h_1},\quad v=\frac{h_3h_1}{h_2},\quad w=\frac{h_1h_2}{h_3}.$$

10. We are now in a position to determine the value of q from the equation $q\,.\,d\rho=d\sigma\,.\,q$, where

$$d\sigma=i\frac{d\xi}{h_1}+j\frac{d\eta}{h_2}+k\frac{d\zeta}{h_3},$$

and

$$d\rho=idx+jdy+kdz.$$

If we write $q=f+il+jm+kn$, we have the equations

$$l\frac{d\xi}{h_1}+m\frac{d\eta}{h_2}+n\frac{d\zeta}{h_3}=ldx+mdy+ndz,$$

$$f\frac{d\xi}{h_1}+n\frac{d\eta}{h_2}-m\frac{d\zeta}{h_3}=fdx+mdz-ndy,$$

$$f\frac{d\eta}{h_2}+l\frac{d\zeta}{h_3}-n\frac{d\xi}{h_1}=fdy+ndx-ldz,$$

$$f\frac{d\zeta}{h_3}+m\frac{d\xi}{h_1}-l\frac{d\eta}{h_2}=fdz+ldy-mdx.$$

The first of these equations may be deduced from the other three, and may consequently be left out of account. The other three must be satisfied independently of the values of the ratios $dx:dy:dz$. This gives rise to the three groups of equations:—

$$\left.\begin{aligned}\frac{f}{h_1}\frac{\partial\xi}{\partial x}+\frac{n}{h_2}\frac{\partial\eta}{\partial x}-\frac{m}{h_3}\frac{\partial\zeta}{\partial x}&=f\\ \frac{f}{h_1}\frac{\partial\xi}{\partial y}+\frac{n}{h_2}\frac{\partial\eta}{\partial y}-\frac{m}{h_3}\frac{\partial\zeta}{\partial y}&=-n\\ \frac{f}{h_1}\frac{\partial\xi}{\partial z}+\frac{n}{h_2}\frac{\partial\eta}{\partial z}-\frac{m}{h_3}\frac{\partial\zeta}{\partial z}&=m\end{aligned}\right\}\dots\dots\dots\dots\text{(A)},$$

$$\left.\begin{aligned}\frac{f}{h_2}\frac{\partial\eta}{\partial x}+\frac{l}{h_3}\frac{\partial\zeta}{\partial x}-\frac{n}{h_1}\frac{\partial\xi}{\partial x}&=n\\ \frac{f}{h_2}\frac{\partial\eta}{\partial y}+\frac{l}{h_3}\frac{\partial\zeta}{\partial y}-\frac{n}{h_1}\frac{\partial\xi}{\partial y}&=f\\ \frac{f}{h_2}\frac{\partial\eta}{\partial z}+\frac{l}{h_3}\frac{\partial\zeta}{\partial z}-\frac{n}{h_1}\frac{\partial\xi}{\partial z}&=-l\end{aligned}\right\}\dots\dots\dots\dots\text{(B)},$$

$$\left.\begin{aligned}\frac{f}{h_3}\frac{\partial\zeta}{\partial x}+\frac{m}{h_1}\frac{\partial\xi}{\partial x}-\frac{l}{h_2}\frac{\partial\eta}{\partial x}&=-m\\ \frac{f}{h_3}\frac{\partial\zeta}{\partial y}+\frac{m}{h_1}\frac{\partial\xi}{\partial y}-\frac{l}{h_2}\frac{\partial\eta}{\partial y}&=l\\ \frac{f}{h_3}\frac{\partial\zeta}{\partial z}+\frac{m}{h_1}\frac{\partial\xi}{\partial z}-\frac{l}{h_2}\frac{\partial\eta}{\partial z}&=f\end{aligned}\right\}\dots\dots\dots\dots\text{(C)}$$

These nine equations between the four ratios $f:l:m:n$, imply six relations, which may be shewn to be equivalent to each of the groups at the beginning of Art. 9.

From the first two equations of group (A) we obtain

$$\frac{n}{f}\left\{\frac{1}{h_2h_3}\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial y}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\left(1+\frac{1}{h_2}\frac{\partial\eta}{\partial y}\right)\right\}=\frac{1}{h_3}\frac{\partial\zeta}{\partial y}\left(1-\frac{1}{h_1}\frac{\partial\xi}{\partial x}\right)+\frac{1}{h_3h_1}\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial y},$$

i.e.
$$\frac{n}{f}\left\{\frac{1}{h_2h_3}\frac{\partial(\eta,\zeta)}{\partial(x,y)}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}=\frac{1}{h_3h_1}\frac{\partial(\zeta,\xi)}{\partial(x,y)}+\frac{1}{h_3}\frac{\partial\zeta}{\partial y},$$

or
$$n\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}=f\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial z}+\frac{1}{h_3}\frac{\partial\zeta}{\partial y}\right\}.$$

Similarly from the first and third equations of the same group we should obtain

$$m\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}=f\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial z}+\frac{1}{h_3}\frac{\partial\zeta}{\partial y}\right\}.$$

The groups (B) and (C) could be treated in the same manner, and thus we should obtain

$$f\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}+\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}=m\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}=n\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\},$$

$$f\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}+\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}=n\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}-\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}=l\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\},$$

$$f\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}+\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}=l\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}=m\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}-\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}.$$

These equations imply

$$\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial y}\right)^2-\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial z}\right)^2=\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial z}\right)^2-\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial x}\right)^2=\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial x}\right)^2-\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial y}\right)^2,$$

which may be readily verified from the equations at the commencement of Art. 9. They also yield the relations

$$\frac{l}{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}-\frac{1}{h_2}\frac{\partial\eta}{\partial z}}=\frac{m}{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}}=\frac{n}{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}}.$$

Further, we have

$$f\left\{\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial y}\right)^2-\frac{1}{h_2^2}\left(\frac{\partial\eta}{\partial z}\right)^2\right\}=f\left\{\frac{1}{h_1^2}\left(\frac{\partial\xi}{\partial z}\right)^2-\frac{1}{h_3^2}\left(\frac{\partial\zeta}{\partial x}\right)^2\right\}$$
$$=l\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}.$$

Now

$$\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}$$
$$=\frac{1}{h_1h_2}\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial x}-\frac{1}{h_1^2}\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}-\frac{1}{h_2h_3}\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{1}{h_3h_1}\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial y}$$
$$=\frac{1}{h_1h_2}\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial x}+\frac{1}{h_2^2}\frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial z}+\frac{1}{h_3^2}\frac{\partial\zeta}{\partial y}\frac{\partial\zeta}{\partial z}+\frac{1}{h_2h_3}\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}$$
$$+\frac{1}{h_2h_3}\frac{\partial\eta}{\partial z}\frac{\partial\zeta}{\partial z}+\frac{1}{h_3h_1}\frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial y}$$
$$=\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}+\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial y}+\frac{1}{h_3}\frac{\partial\zeta}{\partial z}\right\}+\frac{1}{h_3}\frac{\partial\zeta}{\partial y}$$
$$+\frac{1}{h_1h_2}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial z}+\frac{1}{h_2}\frac{\partial\eta}{\partial z}+\frac{1}{h_3h_1}\frac{\partial\zeta}{\partial y}\frac{\partial\xi}{\partial x}$$
$$=\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}+\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}\left\{1+\frac{1}{h_1}\frac{\partial\xi}{\partial x}+\frac{1}{h_2}\frac{\partial\eta}{\partial y}+\frac{1}{h_3}\frac{\partial\zeta}{\partial z}\right\},$$

Therefore

$$\frac{l}{\dfrac{1}{h_3}\dfrac{\partial\zeta}{\partial y}-\dfrac{1}{h_2}\dfrac{\partial\eta}{\partial z}}=\frac{f}{1+\dfrac{1}{h_1}\dfrac{\partial\xi}{\partial x}+\dfrac{1}{h_2}\dfrac{\partial\eta}{\partial y}+\dfrac{1}{h_3}\dfrac{\partial\zeta}{\partial z}};$$

and, we obtain

$$q=1+\frac{1}{h_1}\frac{\partial\xi}{\partial x}+\frac{1}{h_2}\frac{\partial\eta}{\partial y}+\frac{1}{h_3}\frac{\partial\zeta}{\partial z}+i\left\{\frac{1}{h_3}\frac{\partial\zeta}{\partial y}-\frac{1}{h_2}\frac{\partial\eta}{\partial z}\right\}$$
$$+j\left\{\frac{1}{h_1}\frac{\partial\xi}{\partial z}-\frac{1}{h_3}\frac{\partial\zeta}{\partial x}\right\}+k\left\{\frac{1}{h_2}\frac{\partial\eta}{\partial x}-\frac{1}{h_1}\frac{\partial\xi}{\partial y}\right\}.$$

11. Suppose that we have a second orthogonal system of surfaces, and let α, β, γ be the parameters of the three families of the system. Also let

$$\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2=k_1^2,$$
$$\left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2=k_2^2,$$
$$\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2+\left(\frac{\partial\gamma}{\partial z}\right)^2=k_3^2.$$

Then, if we express the equations belonging to our former system in terms of the elements of this system, we obtain

$$k_1^2\left(\frac{\partial\xi}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\xi}{\partial\beta}\right)^2+k_3^2\left(\frac{\partial\xi}{\partial\gamma}\right)^2=h_1^2,$$
$$k_1^2\left(\frac{\partial\eta}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\eta}{\partial\beta}\right)^2+k_3^2\left(\frac{\partial\eta}{\partial\gamma}\right)^2=h_2^2,$$
$$k_1^2\left(\frac{\partial\zeta}{\partial\alpha}\right)^2+k_2^2\left(\frac{\partial\zeta}{\partial\beta}\right)^2+k_3^2\left(\frac{\partial\zeta}{\partial\gamma}\right)^2=h_3^2,$$
$$k_1^2\frac{\partial\eta}{\partial\alpha}\frac{\partial\zeta}{\partial\alpha}+k_2^2\frac{\partial\eta}{\partial\beta}\frac{\partial\zeta}{\partial\beta}+k_3^2\frac{\partial\eta}{\partial\gamma}\frac{\partial\zeta}{\partial\gamma}=0,$$
$$k_1^2\frac{\partial\zeta}{\partial\alpha}\frac{\partial\xi}{\partial\alpha}+k_2^2\frac{\partial\zeta}{\partial\beta}\frac{\partial\xi}{\partial\beta}+k_3^2\frac{\partial\zeta}{\partial\gamma}\frac{\partial\xi}{\partial\gamma}=0,$$
$$k_1^2\frac{\partial\xi}{\partial\alpha}\frac{\partial\eta}{\partial\alpha}+k_2^2\frac{\partial\xi}{\partial\beta}\frac{\partial\eta}{\partial\beta}+k_3^2\frac{\partial\xi}{\partial\gamma}\frac{\partial\eta}{\partial\gamma}=0.$$

Treating these equations by a similar method to that adopted in Art. 9, we obtain

$$\frac{k_1 \dfrac{\partial\xi}{\partial\alpha}}{k_2k_3 \dfrac{\partial(\eta,\zeta)}{\partial(\beta,\gamma)}} = \frac{k_2 \dfrac{\partial\xi}{\partial\beta}}{k_3k_1 \dfrac{\partial(\eta,\zeta)}{\partial(\gamma,\alpha)}} = \frac{k_3 \dfrac{\partial\xi}{\partial\gamma}}{k_1k_2 \dfrac{\partial(\eta,\zeta)}{\partial(\alpha,\beta)}} = \frac{h_1}{h_2h_3},$$

$$\frac{k_1 \dfrac{\partial\eta}{\partial\alpha}}{k_2k_3 \dfrac{\partial(\zeta,\xi)}{\partial(\beta,\gamma)}} = \frac{k_2 \dfrac{\partial\eta}{\partial\beta}}{k_3k_1 \dfrac{\partial(\zeta,\xi)}{\partial(\gamma,\alpha)}} = \frac{k_3 \dfrac{\partial\eta}{\partial\gamma}}{k_1k_2 \dfrac{\partial(\zeta,\xi)}{\partial(\alpha,\beta)}} = \frac{h_2}{h_3h_1},$$

$$\frac{k_1 \dfrac{\partial\zeta}{\partial\alpha}}{k_2k_3 \dfrac{\partial(\xi,\eta)}{\partial(\beta,\gamma)}} = \frac{k_2 \dfrac{\partial\zeta}{\partial\beta}}{k_3k_1 \dfrac{\partial(\xi,\eta)}{\partial(\gamma,\alpha)}} = \frac{k_3 \dfrac{\partial\zeta}{\partial\gamma}}{k_1k_2 \dfrac{\partial(\xi,\eta)}{\partial(\alpha,\beta)}} = \frac{h_3}{h_1h_2}.$$

If we write $$d\sigma' = i\frac{d\alpha}{k_1} + j\frac{d\beta}{k_2} + k\frac{d\gamma}{k_3},$$

we shall have an equation of the form $d\sigma' = p \,.\, d\rho \,.\, p^{-1}$, where p is some quaternion. Combining this equation with the equation $d\sigma = q \,.\, d\rho \,.\, q^{-1}$, we obtain

$$d\sigma = qp^{-1} \,.\, d\sigma' \,.\, pq^{-1};$$

and, if we write $Q = qp^{-1}$, this equation becomes $d\sigma = Q \,.\, d\sigma' \,.\, Q^{-1}$. Utilizing a similar method to that adopted in the preceding article, we shall at length obtain

$$Q = 1 + \frac{k_1}{h_1}\frac{\partial\xi}{\partial\alpha} + \frac{k_2}{h_2}\frac{\partial\eta}{\partial\beta} + \frac{k_3}{h_3}\frac{\partial\zeta}{\partial\gamma} + i\left\{\frac{k_2}{h_3}\frac{\partial\zeta}{\partial\beta} - \frac{k_3}{h_2}\frac{\partial\eta}{\partial\gamma}\right\}$$
$$+ j\left\{\frac{k_3}{h_1}\frac{\partial\xi}{\partial\gamma} - \frac{k_1}{h_3}\frac{\partial\zeta}{\partial\alpha}\right\} + k\left\{\frac{k_1}{h_2}\frac{\partial\eta}{\partial\alpha} - \frac{k_2}{h_1}\frac{\partial\xi}{\partial\beta}\right\}.$$

June 4, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

E. H. GRIFFITHS, M.A., Sidney Sussex College, was elected a Fellow of the Society.

The following Exhibitions and Communications were made:

(1) *Exhibition of Steropus madidus and Earthworm.* By Professor HUGHES.

The author exhibited a specimen of the common earthworm and of the larva of Steropus madidus which was attacking it.

The earthworm was noticed crawling out of the earth and retreating in an irregular manner that roused suspicion. On turning up the earth containing the worm hole and opening it along the line of the burrow an enlarged portion or chamber was exposed in which a small grub about $\frac{3}{4}$ inch long, enveloped in slime discharged by the wounded worm, still held on to it about $1\frac{1}{2}$ inch from the tail. This worm was not yet much injured but in a case previously observed the last inch-and-a-half was nearly severed. What struck the observer most was the apparent paralysis of the worm when seized by the larva. Though about 100 times as large and half-way out of the ground, the worm appeared to be firmly held and either forced to retract itself by the pain of the wound or actually dragged back into the hole where it was at the mercy of its enemy.

(2) *Exhibition of blue-green decayed wood.* By Professor Hughes.

The author exhibited some specimens of blue-green decayed wood in which the colouring matter had penetrated to the heart of the wood though the wood was still so sound that it could be sawn across. Whatever the colouring matter might be it was capable of being worked up by wasps in making their paper nests. He quoted several different though perhaps with some modifications not all of them contradictory opinions as to the origin of the colour, (1) that it was directly due to a microscopic fungus, (2) that it was indirectly caused by a saprophytic organism but was due to the carrying of the colouring matter from that organism by moisture along the tissue of the wood, (3) that it was an organic salt of iron, the organic acid being derived from the decomposing woody tissue and therefore not necessarily connected with any secondary growth.

He asked for further information as to the mode of diffusion of the colouring matter seeing that all circulation must have ceased in the dead wood; and also as to the cells or tissues affected by it.

(3) *Note on Beekite.* By Professor Hughes.

The author explained the mode of occurrence of the form of chalcedony known as Beekite, and pointed out that the usual description, viz. that it was deposited on fossil organisms, was not quite correct, as it replaced portions of the limestone included in the New Red brecciated conglomerates whether they contained fossils or not. The only influencing condition appeared to be the presence, a greater or smaller quantity, of organic matter in the limestone, as he observed that it was more apt to occur on the fragments of bituminous limestone than on those of the more altered crystalline rock.

The next point to which he drew attention was that, whereas in the more decomposed part of the New Red where water had freely percolated, the limestone was often coated all round by Beekite and sometimes upon the decomposition of the whole of the limestone only a chalcedonic shell remained in which the earthy residuum rattled when shaken and which was so thin that it would float in water, on the other hand in the more solid part of the conglomerate the Beekite was apt to occur on the upper surface only and frequently was seen only on the part of the specimen washed by the spray. It seemed improbable that chalcedony could have been thrown down since these pebbles were exposed to the action of surface water; and when, seeking an explanation, he had selected specimens thus partly developed, he noticed that the Beekite stood out from the surface as far only as the limestone pebble had originally extended. Thus it was suggested that the Beekite was formed within the limestone fragments and only developed by the removal of the portion of the rock which had not been silicified. In order to put this to the test of experiment he had broken some of the solid fragments and found that the exterior was silicified. Slices of this exterior showed under the microscope the incipient concentric arrangement of the chalcedony, and when pebbles which showed no trace of Beekite were left for a short time in dilute hydrochloric acid the Beekite was developed on the under side or other parts of the pebble where it had not been previously exposed.

The Beekite was therefore only chalcedonic chert formed at some unknown time and under conditions not yet ascertained *in* the partially silicified exterior of the fragments of limestone, and subsequently developed upon the removal of the calcareous portion after the exposure of the rock in which they occur to the action of surface water.

He exhibited examples of similar formations from Carboniferous, Jurassic, and Cretaceous rocks.

(4) *Exhibition of a series of Photo-micrographs in illustration of the Radulae of Mollusca, and of Apparatus used in photographing the same.* By A. H. Cooke, M.A., King's College.

The author explained the construction and working of apparatus for photo-micrography, and exhibited a series of about one hundred photographs of the Radulae of Mollusca, taken from specimens in Mr H. M. Gwatkin's collection.

(5) *On Variations of Cardium edule from Lagoons in the Nile Delta.* By W. Bateson, M.A., St John's College.

(6) *Preliminary note on the Germination of Seeds.* By M. C. Potter, M.A., St Peter's College.

(7) *On the waves on a viscous rotating cylinder, an illustration of the influence of viscosity on the stability of rotating liquid.* By G. H. Bryan, B.A., St Peter's College.

1. In his important memoir on the stability of rotating liquid*, M. Poincaré has proved that under certain circumstances Maclaurin's spheroid, if formed of a perfect liquid, may be stable even though the total kinetic and potential energy for given angular momentum is not a minimum. If however there be any viscosity in the liquid such figures of relative equilibrium must be unstable.

Since then the conditions of stability of rotating liquid are affected by the presence or absence of viscosity, I thought it might be of interest to give a hydrodynamical investigation of the waves on viscous rotating liquid, in some simple case which admits of mathematical solution, with the view of showing more clearly what is the effect of viscosity on the stability of the relative equilibrium.

Now this can be done if the motion is two dimensional, the surface of the liquid being compelled to remain cylindrical by constraints which do not otherwise affect the motion. Without some such constraints the cylinder would obviously be highly unstable. In the following investigation the cylinder is supposed to be oscillating about steady rotation in the circular form, this being apparently the only case in which the solution leads to intelligible results. In the simple sub-case when the liquid is not rotating the problem becomes exactly analogous, in two dimensions, to Prof. H. Lamb's investigation for the oscillations about the spherical form†.

2. We suppose the cylinder when undisturbed to be of radius a and to rotate about its axis with angular velocity ω. The liquid is supposed self-attracting, of density ρ, the *kinematical* coefficient of viscosity being represented by ν and the surface tension by T.

Let the motion be referred to rectangular axes of x, y rotating about the axis of the cylinder with angular velocity ω. When the liquid is steadily rotating the coordinates (x, y) of any fluid particle referred to these axes will remain constant. Hence when the liquid is slightly disturbed the rates of change of these coordinates will be small. Let them be denoted by u, v. Then the total component velocities of the particle parallel to the instantaneous positions of the axes are $u - \omega y$ and $v + \omega x$.

* "Sur l'équilibre d'une masse fluide animée d'une mouvement de rotation." *Acta mathematica*, Vol. VII. p. 259.

† "On the Oscillations of a Viscous Spheroid." *Proc. London Math. Soc.* Vol. XIII. p. 51.

By the method of Greenhill*, the fundamental equations of Hydrodynamics for this two dimensional motion of viscous liquid are easily shown to be

$$\frac{\partial u}{\partial t} - \omega(v + \omega x) + u\frac{\partial u}{\partial x} + v\left(\frac{\partial u}{\partial y} - \omega\right) - \nu\nabla^2 u = \frac{\partial}{\partial x}\left(V - \frac{p}{\rho}\right) \ldots (1),$$

$$\frac{\partial v}{\partial t} + \omega(u - \omega y) + u\left(\frac{\partial v}{\partial x} + \omega\right) + v\frac{\partial v}{\partial y} - \nu\nabla^2 v = \frac{\partial}{\partial y}\left(V - \frac{p}{\rho}\right) \ldots (2),$$

while the equation of continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

V being the potential due to attraction of the liquid or other causes, and p the *mean* pressure about the point (x, y).

From (3) it appears that there is a function ψ, such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \ldots\ldots\ldots\ldots\ldots (4).$$

Let us neglect squares and products of u, v and write

$$\varpi = V - \frac{p}{\rho} + \tfrac{1}{2}\omega^2(x^2 + y^2) \ldots\ldots\ldots\ldots (5),$$

then the equations (1), (2) become

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{\partial \psi}{\partial y} + 2\omega\frac{\partial \psi}{\partial x} &= \frac{\partial \varpi}{\partial x} \\ \left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{\partial \psi}{\partial x} - 2\omega\frac{\partial \psi}{\partial y} &= -\frac{\partial \varpi}{\partial y} \end{aligned}\right\} \ldots\ldots\ldots (6).$$

Differentiating with respect to y, x respectively, and adding we find

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\psi = 0 \ldots\ldots\ldots\ldots\ldots (7).$$

3. Let us assume that ψ (and therefore ϖ) $\propto e^{-\alpha t}$, where α is complex.

Transform to cylindrical (polar) coordinates, (r, θ). A solution of (7), which does not become infinite at the origin, is

$$\psi = e^{-\alpha t} e^{\iota n\theta}\{Ar^n + BJ_n(hr)\} \ldots\ldots\ldots\ldots (8),$$

provided $$h^2 = \alpha/\nu \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

J_n denoting as usual Bessel's function of order n and A, B, being arbitrary constants.

* *Encyclopædia Britannica*, article "Hydromechanics."

Also the equations (6) when transformed to polars lead to

$$\left.\begin{aligned} \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\psi+2\omega r\frac{\partial\psi}{\partial r}&=r\frac{\partial\varpi}{\partial r}\\ r\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\psi-2\omega\frac{\partial\psi}{\partial\theta}&=-\frac{\partial\varpi}{\partial\theta}\end{aligned}\right\}\text{............(10)},$$

from which we find

$$\varpi=e^{-\alpha t}e^{\iota n\theta}\{(2\omega-\iota\alpha)Ar^n+2\omega BJ_n(hr)\}\text{........ (11)}.$$

4. The equation of the free surface is

$$r=a+ce^{\iota n\theta}\text{............................(12)},$$

where

$$\frac{\partial ce^{\iota n\theta}}{\partial t}=\text{normal component velocity of liquid}=\left(\frac{\partial\psi}{r\partial\theta}\right)_{r=a}$$

$$\therefore\ c=-\frac{\iota n}{\alpha a}e^{-\alpha t}\{Aa^n+BJ_n(ha)\}\text{ ....(13)}.$$

The gravitation potential of the deformed cylinder at any point (r, θ) is to the first power of c

$$V_1=\pi\rho\gamma(a^2-r^2)+2\pi\rho\gamma\frac{ca}{n}\frac{r^n}{a^n}e^{\iota n\theta}+C\text{......... (14)}$$

if the point be inside the cylinder, or

$$V_0=-2\pi\rho\gamma a^2\log\frac{r}{a}+2\pi\rho\gamma\frac{ca}{n}\frac{a^n}{r^n}e^{\iota n\theta}+C\text{......(15)}$$

if the point be outside, γ being the constant of gravitation and C some constant. Thus at any point of the deformed surface the potential is

$$V=-2\pi\rho\gamma ca\left(1-\frac{1}{n}\right)e^{\iota n\theta}+C\text{..............(16)},$$

and $$\begin{aligned}\frac{p}{\rho}&=V+\tfrac{1}{2}\omega^2r^2-\varpi\\ &=-e^{-\alpha t}e^{\iota n\theta}\left[\frac{\iota n}{\alpha}\left\{Aa^n+BJ_n(ha)\right\}\left\{\omega^2-2\pi\rho\gamma\left(1-\frac{1}{n}\right)\right\}\right.\\ &\quad\left.+(2\omega-\iota\alpha)Aa^n+2\omega BJ_n(ha)\right]+C+\tfrac{1}{2}\omega^2a^2\text{ .... (17)}.\end{aligned}$$

If $1/R$ be the curvature at any point of the surface, then it is readily shown that

$$\frac{1}{R}=\frac{1}{a}+(n^2-1)\frac{c}{a^2}e^{\iota n\theta}\text{.................(18)}.$$

5. We may now show how to determine the forced oscillations or waves produced when, in addition to surface tension, small variable disturbing forces act on the surface of the liquid. Let the normal and tangential components of these surface tractions at any point be $F, G,$ and let the radial and transversal component velocities of the fluid *relative* to the system of moving axes be $U, V,$ so that

$$U = \frac{\partial\psi}{r\partial\theta}, \quad V = -\frac{\partial\psi}{\partial r} \quad \text{......................(19).}$$

For the component rates of distortion of the liquid we have*

$$\left.\begin{aligned} &e_1, \text{ (rate of elongation along } r) &&= \frac{\partial U}{\partial r} \\ &e_2, \text{ (rate of elongation perpendicular to } r) &&= \frac{\partial V}{r\partial\theta} + \frac{U}{r} \\ &s, \text{ (the rate of shear)} &&= \frac{\partial U}{r\partial U} + \frac{\partial V}{\partial r} - \frac{V}{r} \end{aligned}\right\} \text{(20),}$$

and for the stresses we get

$$\left.\begin{aligned} &\frac{F}{\rho} - \frac{T}{\rho R} = -\frac{p}{\rho} + 2\nu e_1 \\ &\frac{G}{\rho} = \nu s \end{aligned}\right\} \text{..................(21).}$$

Substituting we find

$$\frac{F}{\rho} - \frac{T}{\rho R} = -\frac{p}{\rho} + 2\nu \frac{\partial}{\partial r}\frac{\partial\psi}{r\partial\theta} \quad \text{..................(22),}$$

$$\frac{G}{\rho} = \nu\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) \quad \text{..............(23),}$$

in which r is put $= a$ after the differentiations have been performed.

Whatever be the expressions for $F, G,$ they may be expanded in circular functions of multiples of θ, and the coefficients which are functions of the time may be expanded in terms of the form $e^{-\alpha t}$ where the α's are certain, real, imaginary or complex constants. Thus, F, G are expressible as series of terms of the form $fe^{-\alpha t}e^{\iota n\theta}$, $ge^{-\alpha t}e^{\iota n\theta}$ where f, g are constants, and the effect of each term must be found separately.

The corresponding expression for ψ will have the same α whilst h will be determined by (9).

* R. R. Webb, "On Stress and Strain in Cylindrical and Polar Coordinates." *Messenger of Mathematics*, Feb. 1882, p. 147.

Substituting these values of F, G and the expressions for ψ and $\frac{1}{R}$ in (22), (23) we get, omitting the constant terms,

$$\frac{f}{\rho} = (2\omega - \iota\alpha) A a^n + 2\omega B J_n(ha)$$
$$+ \frac{\iota n}{\alpha}\{Aa^n + BJ_n(ha)\}\left\{\omega^2 - 2\pi\rho\gamma\frac{n-1}{n} - \frac{T}{\rho}\frac{n^2-1}{a^3}\right\}$$
$$+ \frac{2\nu\iota n}{a^2}\{(n-1)Aa^n + B[haJ_n'(ha) - J_n(ha)]\} \ldots (24),$$

$$\frac{g}{\rho} = -\frac{\nu}{a^2}\{2n(n-1)Aa^n$$
$$+ B[h^2a^2J_n''(ha) - haJ_n'(ha) + n^2J_n(ha)]\} \ldots (25).$$

These two equations determine the two unknown constants A, B as linear functions of the given coefficients f, g.

6. If the waves be *free* we must put $f = 0$ and $g = 0$ in these equations. By means of the well-known relations

$$h^2a^2J_n''(ha) + haJ_n'(ha) - (n^2 - h^2a^2)J_n(ha) = 0 \ldots\ldots (26),$$
$$haJ_n'(ha) = nJ_n(ha) - haJ_{n+1}(ha) \ldots\ldots (27),$$

(25) now gives

$$B = \frac{2n(n-1)Aa^n}{2haJ_n'(ha) - (2n^2 - h^2a^2)J_n(ha)}$$
$$= \frac{2n(n-1)Aa^n}{\{h^2a^2 - 2n(n-1)\}J_n(ha) - 2haJ_{n+1}(ha)} \ldots (28).$$

To simplify (24) let us write

$$k_n = -\omega^2 + 2\pi\rho\gamma\frac{n-1}{n} + \frac{T}{\rho}\frac{n^2-1}{a^3} \ldots\ldots (29)$$

and multiply throughout by $\iota\alpha$. We thus obtain

$$Aa^n\left\{\alpha(\alpha + 2\omega\iota) + nk_n - \frac{2\nu}{a^2}n(n-1)\alpha\right\}$$
$$+ B\left\{\left(2\omega\iota\alpha + nk_n - \frac{2\nu}{a^2}n(n-1)\alpha\right)J_n(ha) + \frac{2\nu}{a^2}naha\,J_{n+1}(ha)\right\} = 0$$
$$\ldots\ldots (30).$$

By eliminating the ratio of A to B from (28), (30), we obtain a relation which reduces to

$$\left\{\alpha(\alpha + 2\omega\iota) + nk_n - \frac{4\nu}{a^2}n(n-1)\alpha\right\}\{haJ_n(ha) - 2J_{n+1}(ha)\}$$
$$+ \frac{4\nu}{a^2}n(n-1)^2\alpha J_{n+1}(ha) = 0 \ldots (31).$$

This may be regarded as an equation in h or in α whose roots are the values of either of these quantities corresponding to the various free waves on the cylinder.

If we write
$$z = h^2a^2 = \frac{\alpha a^2}{\nu} \quad \text{.........................(32)},$$

and multiply throughout by $z^{-\frac{n+1}{2}}$, it reduces to an equation in z, viz.

$$\left\{z\left(z + 2\frac{\omega a^2}{\nu}\iota\right) + nk_n\frac{a^4}{\nu^2} - 4n(n-1)z\right\}\left(z^{-n/2}J_n\sqrt{z} - 2z^{-(n+1)/2}J_{n+1}\sqrt{z}\right)$$
$$+ 4n(n-1)^2z \, . \, z^{-(n+1)/2}J_{n+1}\sqrt{z} = 0 \quad \text{......(33)}.$$

$J_{n+1}\sqrt{z}$ is always divisible by $z^{(n+1)/2}$, and we are justified in removing this factor from the equation in z on the grounds that the value $z=0$ is in general inadmissible, as we shall now show.

7. For $z=0$ gives $h=0$, $\alpha=0$. Also when h is indefinitely diminished $J_n(ha)$ becomes ultimately $= h^na^n/\{2^n\Gamma(n+1)\}$. We thus obtain from (28)

$$B = -2^n\Gamma(n+1)A/h^n,$$

and therefore
$$\psi = 0.$$

Hence there is no motion of the fluid—this is as we should expect. If however we suppose that the surface is deformed so that the normal displacement is $ce^{\iota n\theta}$ we find by (24) that the normal surface traction necessary to maintain this displacement and acting outwards is $fe^{\iota n\theta}$ where

$$\frac{f}{\rho} = cak_n,$$

and therefore the displacement will not continue without the application of force unless $k_n=0$. But in this case 0 is a root of the *divided* equation (33). Thus the factor $z^{(n+1)/2}$ is irrelevant.

8. The equation (33) has an infinite number of roots, in general complex, and corresponding to each of these values of z we find a different function ψ given by (8), (28) which satisfies all the conditions of the problem but which is complex. From each of these we may however form an expression ψ corresponding to a real wave motion of the liquid. Assuming z and therefore α and h complex let us write $\alpha = \alpha_1 + \iota\alpha_2$, and express

$$2n(n-1)J_n(hr)/[\{h^2a^2 - 2n(n-1)\}J_n(ha) - 2haJ_{n+1}(ha)]$$

in the form
$$R_1 + \iota R_2,$$

R_1, R_2 being real functions of r. Then the value of ψ in (8) can be written

$$\psi = \frac{A}{a^n} e^{\iota n\theta} \, exp - (\alpha_1 + \iota\alpha_2) t \, . \, \{r^n/a^n + (R_1 + \iota R_2)\}$$

$$= A_1 e^{-\alpha_1 t} \{(r^n/a^n + R_1)^2 + R_2^2\}^{\frac{1}{2}} \, exp \, \iota \left\{ n\theta - \alpha_2 t + \tan^{-1} \frac{R_2}{R_1 + r^n/a^n} \right\} \quad (34)$$

where A_1 is a constant $= A/a^n$.

If we had taken

$$\psi = e^{-\alpha' t} e^{-\iota n\theta} \{A' r^n + B' J_n (h' r)\} \quad \text{.........(35)},$$

we should have obtained instead of (33) the corresponding equation in z

$$\left\{ z\left(z - \frac{2\omega a^2}{\nu}\iota\right) + nk_n \frac{a^4}{\nu^2} - 4n(n-1)z \right\} (z^{-n/2} J_n \sqrt{z} - 2z^{-(n+1)/2} J_{n+1} \sqrt{z})$$
$$+ 4n(n-1)^2 z \, . \, z^{-(n+1)/2} J_{n+1} \sqrt{z} = 0 \quad \text{...........(36)},$$

which may be obtained from (33) by writing $-\omega\iota$ for $\omega\iota$.

The complex roots of this equation are conjugate to those of (33), and this is also true of the values of α' h'. Moreover taking $\alpha' = \alpha_1 - \iota\alpha_2$ we are led to the corresponding solution

$$\psi = A_1 e^{-\iota n\theta} \, exp - (\alpha_1 - \iota\alpha_2) \, t \, . \, \{r^n/a^n + (R_1 - \iota R_2)\}$$

$$= A_1 e^{-\alpha t} \{(r^n/a^n + R_1)^2 + R_2^2\}^{\frac{1}{2}} \, exp - \iota \left\{ n\theta - \alpha_2 t + \tan^{-1} \frac{R_2}{R_1 + r^n/a^n} \right\}$$
$$\text{.............................(37)},$$

and combining (34), (37) we get the real wave motion

$$\psi = A_1 e^{-\alpha_1 t} \{(r^n/a^n + R_1)^2 + R_2\}^{\frac{1}{2}} \begin{matrix} \sin \\ \text{or} \cos \end{matrix} \left\{ n\theta - \alpha_2 t + \tan^{-1} \frac{R_2}{R_1 + r^n/a^n} \right\}$$
$$\text{..........................(38)}.$$

This represents a system of waves travelling round the cylinder with angular velocity α_2/n relatively to the rotating mass or $\omega + \alpha_2/n$ relatively to axes fixed in space, and, if α_1 is positive, gradually dying away, the modulus of decay being $1/\alpha_1$. If α_1 is negative they increase indefinitely with the time till the motion is so large that the squares of the relative velocities of the fluid can be no longer neglected, consequently the circular form of the cylinder is unstable.

The particles of fluid which in the steady motion form a concentric circle of radius r will be disturbed in waves of which the phase is a function of r, and therefore changes as we proceed from the free surface of the cylinder inwards.

The radial velocity of the fluid in these undulations is given by $\frac{\partial\psi}{r\partial\theta}$. Thus its amplitude, and therefore also that of the radial displacement, are, at any time, everywhere proportional to

$$\{(r^n/a^n+R_1)^2+R_2^2\}^{\frac{1}{2}}/r,$$

i.e. to
$$\frac{1}{r}\bmod\{Ar^n+BJ_n(hr)\}.$$

When r is small this becomes approximately proportional to r^{n-1}. Now the slow motions corresponding to $n=0$ or 1 are not waves at all. Excluding these we see that the displacements of the fluid particles diminish indefinitely as r approaches zero, and, the greater be the number of corrugations (n), the more rapidly do they become insensible towards the centre.

9. We now proceed to discuss a few results which hold with regard to the fundamental equation for z

$$\{z(z+2\omega a^2\iota/\nu)+nk_na^4/\nu^2-4n(n-1)z\}\left(z^{-n/2}J_n\sqrt{z}-2z^{-(n+1)/2}J_{n+1}\sqrt{z}\right)$$
$$+4n(n-1)^2z\,.\,z^{-(n+1)/2}J_{n+1}\sqrt{z}=0 \quad \text{.........(33).}$$

It will be necessary to remember* that the roots of the equation
$$z^{-n/2}J_n\sqrt{z}=0 \quad \text{.....................(39)}$$
are all real and positive, and are separated by those of
$$z^{-(n+1)/2}J_{n+1}\sqrt{z}=0 \quad \text{....................(40).}$$

(i) The equation (33) has an infinite number of roots, but since, in it, $z^{-n/2}J_n\sqrt{z}$ occurs multiplied by a quadratic function of z, it has two and only two more roots than has the equation (39).

(ii) The equation (33) cannot have a real root other than zero unless $\omega=0$.

For if we suppose z real and ω not $=0$ and equate the real and imaginary parts of (33) to zero we find that unless $z=0$ we must have
$$z^{-n/2}J_n\sqrt{z}-2z^{-(n+1)/2}J_{n+1}\sqrt{z}=0 \quad \text{...........(41),}$$
and also
$$z^{-(n+1)/2}J_{n+1}\sqrt{z}=0 \quad \text{...................(40),}$$
whence also
$$z^{-n/2}J_n\sqrt{z}=0 \quad \text{.......................(39),}$$
which is impossible since the equations (39), (40) cannot have a common root.

* Lommel, *Studien über die Bessel'schen Functionen*, p. 68.

This shows that if the liquid be rotating there cannot be a system of gradually diminishing waves whose position relative to the rotating mass remains fixed.

(iii) Nor can it have a pair of conjugate complex roots, except when $\omega = 0$.

Since the roots of (36) are conjugate to those of (33) such a pair, if it existed, would have to be roots of both these equations and therefore also of the two equations (39), (40).

Hence we cannot have two waves travelling with equal relative velocities in opposite directions, or combining to form relatively stationary oscillatory motions of the liquid.

(iv) In the cases when $n = 0$ or $n = 1$, if we remove extraneous factors the equation in z reduces to

$$z^{-n/2} J_n \sqrt{z} - 2z^{-(n+1)/2} J_{n+1} \sqrt{z} = 0 \quad \text{............(41)}$$

where $n = 0$ or 1 respectively. Now whatever be the value of n the roots of this equation are all real and positive. For if we put z equal, in turn, to zero, and the successive roots of the equation (39), we find the left-hand side is alternately positive and negative, hence the equation has one root between each of these values of z. We may similarly show that its roots are also separated by those of (40).

When $n = 0$ the motion is symmetrical about the axis of the cylinder, and the fluid particles rotate round this axis with small angular velocities relative to the moving axes, which are functions of the distance from the centre.

When $n = 1$, we get unsymmetrical types of slow motion of the liquid within the cylinder, in which the form of the surface remains unaltered.

In either case the motions gradually die away, and are unaffected by the rotation.

(v) When $\omega = 0$ the equation becomes

$$\{z^2 + nk_n a^4/\nu^2 - 4n(n-1)z\}(z^{-n/2} J_n \sqrt{z} - 2z^{-(n+1)/2} J_{n+1} \sqrt{z}) \\ + 4n(n-1)^2 z \, . \, z^{-(n+1)/2} J_{n+1} \sqrt{z} = 0 \text{...(42)},$$

in which all the coefficients are real. Putting z equal in turn to the various roots of (41) we find by the results just obtained in (i) that the sign of the left hand is alternately positive and negative, hence the equation has at least one real root between each of these values of z. It can therefore at most have two conjugate complex roots, and it may have all its roots real. We shall see

hereafter that the former is the case if the liquid be but slightly viscous, the latter if the viscosity is exceedingly great.

10. It is physically impossible that the equation (33) should be satisfied by a purely imaginary value of z, and therefore of α except when $\nu = 0$. In fact a strictly periodic motion can never exist in the case of viscous liquid unless energy be supplied from without. For after a complete period the liquid would return to its original state, and its energy would therefore be the same as at the beginning. But all motions of viscous liquids except rigid body displacements are accompanied by dissipation of energy which is converted into heat. Such energy, since it is not derived from the system, would have to be supplied from external sources as, for example, in the case of periodic forced waves.

This does not prevent the possibility that the real part of z or α may be negative, provided that the total energy for given angular momentum be not a minimum in the state of relative equilibrium. In this case the dissipated energy will be derived from the system, which will pass into configurations in which the total kinetic and potential energy is less than in the original state.

From this it is evident that in any case of rotating viscous liquid, if we refer the motion to axes rotating with the fluid mass and suppose the small displacements of the liquid particles to vary as $e^{-\alpha t}$, the real part of α can only change from positive to negative when α vanishes. The waves then become stationary corrugations on the surface, and for such displacements the total energy ceases to be a minimum. Accordingly if one of the waves becomes unstable, this must happen when the energy criterion for secular stability ceases to be satisfied.

In the case of the circular cylinder one value of α will vanish and change sign only when $k_n = 0$, and it may readily be verified that for minimum energy k_n must be positive. How the conditions of stability are altered by a complete absence of all viscosity will be best seen by considering the case in which the viscosity of the liquid is supposed relatively small.

11. Let us, then, suppose either that the kinematical viscosity ν is very small or else that the radius a of the cylinder is very great so that ν/a^2 is small in comparison with ω or $\sqrt{\gamma\rho}$. Then the roots of (33) may be expanded in powers of ν/a^2. We must either suppose z to be finite and α small, or α finite and z very large.

12. If we adopt the first hypothesis we find that the values of z are ultimately given by the equation

$$z^{-n/2} J_n \sqrt{z} - 2z^{-(n+1)/2} J_{n+1} \sqrt{z} = 0 \ldots\ldots\ldots\ldots (41),$$

which has already been investigated. Let z_0 be any root of this equation and let us write z_0+x for z in (33), where x is small. Then we find approximately

$$\tfrac{1}{2}x\left(z_0{}^{-(n+1)/2}J_{n+1}\sqrt{z_0}-2z_0{}^{-(n+2)/2}J_{n+2}\sqrt{z_0}\right)$$
$$=\frac{4n(n-1)^2 z_0 z_0{}^{-(n+1)/2}J_{n+1}\sqrt{z_0}}{nk_n a^4/\nu^2+2\omega\iota z_0 a^2/\nu} \quad\ldots\ldots\ldots\ldots(43).$$

Now z_0 is a root of (41), also we have the well-known relation*

$$2(n+1)J_{n+1}\sqrt{z_0}/\sqrt{z_0}=J_n\sqrt{z_0}+J_{n+2}\sqrt{z_0} \quad\ldots\ldots\ldots\ldots(44),$$

whence

$$z_0{}^{-(n+1)/2}J_{n+1}\sqrt{z_0}-2z_0{}^{-(n+2)/2}J_{n+2}\sqrt{z_0}$$
$$=\left(1-\frac{4n}{z_0}\right)z_0{}^{-(n+1)/2}J_{n+1}\sqrt{z_0} \quad\ldots\ldots(45).$$

Therefore neglecting the fourth power of ν/a^2

$$x=\frac{8n(n-1)^2z_0{}^2}{(z_0-4n)}\,\frac{nk_n\nu^2/a^4-2\omega\iota z_0\nu^3/a^6}{n^2k_n{}^2+4\omega^2z_0{}^2\nu^2/a^4}\quad\ldots\ldots\ldots(46).$$

Now the roots of (41) are all greater than $4n$. Hence the imaginary part of α which is a quantity of order $(\nu/a^2)^4$ is essentially negative. From the results of (ii) § 9, it follows that the real part of α must still be negative even where the viscosity is no longer small. Hence all but two of the wave motions travel round in the negative direction relative to our rotating axes, i.e. rotate more slowly than the liquid, but if ν/a^2 be even moderately small the disturbance will have died away long before its position relative to the liquid mass has changed through an appreciable angle.

Again, since $h^2a^2J_n(ha)-2haJ_{n+1}(ha)$ is of order $(\nu/a^2)^2$, therefore by (28) $Aa^n+BJ_n(ha)$ is also of order $(\nu/a^2)^2$.

Thus the height of the corrugations of the surface is very small compared with the components of relative displacement of the fluid particles in the interior.

Unless therefore the liquid possess considerable viscosity or the cylinder be not very great in diameter, the slow motions corresponding to these roots are not waves at all but small vortex motions in the interior of the liquid which die away very slowly, and can never be unstable, nor increase with the time.

13. In the true waves α remains finite and therefore z becomes very large when $\frac{\nu}{a^2}$ is small. We must therefore use Lommel's

* Lommel, p. 8.

expressions for the development of $J_n\sqrt{z}, J_{n+1}\sqrt{z}$ in negative powers of $\sqrt{z}$*, from which it appears that ultimately

$$\left.\begin{aligned} J_n\sqrt{z} &= \left(\frac{2}{\pi\sqrt{z}}\right)^{\frac{1}{2}}\cos\left(\sqrt{z}-\frac{\pi}{4}-n\frac{\pi}{2}\right) \\ J_{n+1}\sqrt{z} &= \left(\frac{2}{\pi\sqrt{z}}\right)^{\frac{1}{2}}\sin\left(\sqrt{z}-\frac{\pi}{4}-n\frac{\pi}{2}\right) \end{aligned}\right\}\ldots\ldots\ldots\ldots(47).$$

Hence, if we neglect powers of ν/a^2 higher than the first, equation (31) gives the following quadratic for α,

$$\alpha(\alpha+2\omega\iota)+nk_n-\frac{4\nu}{a^2}n(n-1)\alpha=0\ \ldots\ldots\ (48).$$

Solving for α, we find to the first order

$$\alpha=\{\sqrt{\omega^2+nk_n}\pm\omega\}\left\{2\frac{\nu}{a^2}\frac{n(n-1)}{\sqrt{\omega^2+nk_n}}\mp\iota\right\}\ \ldots\ldots\ldots\ldots\ (49),$$

the upper or lower sign being taken throughout.

14. If k_n is positive the real parts of both values of α given by (49) are positive and therefore the waves diminish indefinitely as the time increases. For wave motions of the type considered the cylinder is secularly stable.

If k_n is negative the real part of *one* of the values of α becomes negative and the corresponding wave continually increases until it is no longer small, therefore the cylinder is secularly unstable.

Since the real part of α is proportional to $\frac{\nu}{a^2}$, it follows that the smaller we suppose this quantity the more slowly will the waves increase or diminish.

If the liquid be *perfect* and ω^2+nk_n be positive, the values of α will be purely imaginary. Hence the waves will be strictly periodic and will neither increase nor diminish. The cylindrical form will, for such wave displacements, be stable, but will possess only what M. Poincaré calls "ordinary" stability. There will be no tendency to return to or depart from the undisturbed state of steady rotation and any waves produced by the disturbance will continue permanently.

If however ω^2+nk_n be negative, the values of α will be complex even if $\nu=0$. We now obtain

$$\alpha=-\omega\iota\pm\sqrt{-(\omega^2+nk_n)}\ \ldots\ldots\ldots\ldots\ldots\ldots(50).$$

The real part of one root is positive but that of the other is negative. Hence even if the fluid be inviscid one of the waves

* *Studien*, § 17, p. 57.

will increase indefinitely with the time, and the cylinder will not possess even ordinary stability.

15. The condition for secular stability of the viscous cylinder is that for all integral values of n greater than unity

$$\omega^2 < 2\frac{n-1}{n}\pi\rho\gamma + (n^2-1)\frac{T}{\rho a^3} \qquad (51).$$

The right-hand side is least if $n = 2$, hence we must have

$$\omega^2 < \pi\rho\gamma + 3T/\rho a^3 \qquad (52).$$

For a perfect liquid the condition of ordinary stability is

$$\omega^2 - \frac{\omega^2}{n} < 2\frac{n-1}{n}\pi\rho\gamma + (n^2-1)\frac{T}{\rho a^3},$$

or

$$\omega^2 < 2\pi\rho\gamma + n(n+1)\frac{T}{\rho a^3} \qquad (53).$$

If there be no surface tension the fluid will become unstable for *all* displacements when

$$\omega^2 > 2\pi\rho\gamma \qquad (54).$$

It may readily be shown that when this is the case the liquid could not remain in the circular form unless subjected to external hydrostatic pressure. Without such pressure a hollow would form in the centre.

If T is not $= 0$ the greatest angular velocity is given by

$$\omega^2 = 2\pi\rho\gamma + 6T/\rho a^3 \qquad (55)$$

while in order that external pressure may not be needed

$$\omega^2 < 2\pi\rho\gamma + T/\rho a^3 \qquad (56),$$

this will happen *before* the cylinder ceases to possess ordinary stability. It will be seen from above that the greatest angular velocity consistent with stability is $\sqrt{2}$ times as great for perfect as for viscous liquid.

If we reduce the liquid to rest and replace the centrifugal force by a force $\lambda^2 r$ from the axis, we find if the liquid be perfect

$$\alpha = \pm\iota\sqrt{nk_n} \qquad (57),$$

where

$$k_n = 2\pi\rho\gamma\left(1-\frac{1}{n}\right) + \frac{T}{\rho}\frac{n^2-1}{a^3} - \lambda^2 \qquad (58),$$

so that stability now ceases when $k_n = 0$. When the liquid is rotating, ordinary stability ceases when the equation in α has a pair of *equal* roots, but since the two waves travelling in opposite directions have not the same period this does not happen when

one value of α becomes zero and when, as pointed out in § 10, the cylinder ceases to remain secularly stable. This property applies to rotating liquids in general.

16. If we proceed to the next approximation we find for the equation in α

$$\alpha(\alpha+2\omega\iota)+nk_n-4n(n-1)\alpha\frac{\nu}{a^2}\left\{1-(n-1)\left(\frac{\nu}{a^2}\right)^{\frac{1}{2}}\frac{1}{\iota\sqrt{\alpha}}\right\}=0\ldots(59).$$

Let us write λ_1, λ_2 respectively for the two expressions

$$\sqrt{\omega^2+nk_n}+\omega \text{ and } \sqrt{\omega^2+nk_n}-\omega.$$

The values of α will be given by

$$\alpha=\frac{\nu}{a^2}\frac{n(n-1)}{\sqrt{\omega^2+nk_n}}\{2\lambda_r-(n-1)(\nu/a^2)^{\frac{1}{2}}\sqrt{2\lambda_r}\}$$
$$+\overline{-1}|^r\iota\left\{\lambda_r-\frac{2n(n-1)^2}{\sqrt{\omega^2+nk_n}}\left(\frac{\nu}{a^2}\right)^{\frac{3}{2}}\sqrt{2\lambda_r}\right\}(r=1 \text{ or } 2)\ldots(60),$$

showing that, if we retain terms of order $(\nu/a^2)^{\frac{3}{2}}$, both real and imaginary parts of α will be diminished and therefore the waves will be slightly retarded (relatively) by viscosity.

17. We may also readily obtain first approximations to the values of α when ν/a^2 is very large. This will apply to the cases in which either the liquid is highly viscous or the radius of the cylinder very small.

If we suppose z finite α will be very great and z will be given by

$$\{z-4n(n-1)\}\left(z^{-n/2}J_n\sqrt{z}-2z^{-(n+1)/2}J_{n+1}\sqrt{z}\right)$$
$$+4n(n-1)^2z^{-(n+1)/2}J_{n+1}\sqrt{z}=0\ldots(61),$$

the root $z=0$ being excluded from this approximation. The roots of this equation are real and positive and separated by those of (41). Owing to the largeness of α the relative motions will die away very rapidly, being quickly annulled by viscosity.

If α be not great z will be small. Putting therefore

$$z^{-n/2}J_n\sqrt{z}=\frac{1}{2^n\Gamma(n+1)},$$

$$z^{-(n+1)/2}J_{n+1}\sqrt{z}=\frac{1}{2^{n+1}\Gamma(n+2)},$$

we find for α the quadratic

$$\alpha(\alpha+2\omega\iota)+nk_n-2(n^2-1)\alpha\nu/a^2=0\ \ldots\ldots\ldots(62).$$

When ν/a^2 is large, one root of this equation is large the other being small. Since by hypothesis α is not large, we find approximately

$$\alpha = \frac{nk_n}{2\{(n^2-1)\nu/a^2 - \omega\iota\}} \quad\text{.................(63).}$$

Here the corrugations on the surface die away very slowly owing to the great resistance offered by viscosity to changes of form of the liquid under its attraction and capillarity.

18. If $\omega = 0$ the value of α given by (63) is real. Moreover since the equation (42) does not involve any imaginary quantities in its expression, it is evident that if we proceed to higher approximations in the expansion of the various values of z or α in ascending powers of the small quantity a^2/ν no imaginaries can enter into them.

Hence follows the result stated in (v) § 9, viz. that all the values of z are real provided that their expansions in descending powers of ν/a^2 are convergent. When this is so, the slow motions do not partake of the nature of waves.

Waves on the viscous liquid surrounding a solid rotating cylinder.

19. By introducing Bessel's functions of the second kind the method of this paper may be extended to any problem relating to the two dimensional waves on viscous rotating liquids which, in the state of relative equilibrium, are bounded by concentric cylindrical surfaces. As an example, let us take the case when the liquid contains a perfectly rigid cylindrical nucleus of density σ and radius b, the outer radius of the liquid surface being still a.

We must now assume for ψ the expression

$$\psi = \{Ar^n + A'r^{-n} + BJ_n(hr) + B'Y_n(hr)\}e^{-\alpha t}e^{\iota n\theta} \quad\text{......(64)}$$

involving *four* arbitrary constants.

We find

$$\varpi = \{(2\omega - \iota\alpha)Ar^n + (2\omega + \iota\alpha)A'r^{-n} + 2\omega BJ_n(hr) + 2\omega B'Y_n(hr)\}e^{-\alpha t}e^{\iota n\theta}$$
$$\text{.............................(65).}$$

We suppose that the nucleus is rotating with angular velocity ω, and that no slipping of the liquid takes place at the surface of the solid. Thus $\partial\psi/\partial r$ and $\partial\psi/r\partial\theta$ both vanish when $r = b$, giving the boundary conditions

$$Ab^n + A'b^{-n} + BJ_n(hb) + B'Y_n(hb) = 0 \quad\text{.........(66),}$$

$$nAb^n - nA'b^{-n} + hbBJ_n'(hb) + hbB'Y_n'(hb) = 0 \quad\text{...(67).}$$

The boundary conditions at the free surface $r = a$ are to be found in just the same way as when there is no central nucleus, but the attraction of the nucleus must be taken into account. The values of V_1, V_0 given by (14), (15) will be increased by

$$-2\pi(\sigma-\rho)b^2\log\frac{r}{a},$$

and instead of (29) we shall have to assume

$$k_n = 2\pi\rho\gamma\left\{\frac{n-1}{n}+\frac{b^2}{a^2}\left(\frac{\sigma}{\rho}-1\right)\right\}+(n^2-1)\frac{T}{\rho a^3}-\omega^2 \ldots (68).$$

We shall then obtain, resolving tangentially,

$$\left.\begin{array}{l} 2n(n-1)Aa^n+2n(n+1)A'a^{-n} \\ -B\left[\{h^2a^2-2n(n-1)\}J_n(ha)-2haJ_{n+1}(ha)\right] \\ -B'\left[\{h^2a^2-2n(n-1)\}Y_n(ha)-2haY_{n+1}(ha)\right] \end{array}\right\} = 0 \ldots (69).$$

Resolving normally

$$\left.\begin{array}{l} Aa^n\{\alpha(\alpha+2\omega\iota)+nk_n-2n(n-1)\nu\alpha/a^2\} \\ +A'a^{-n}\{\alpha(-\alpha+2\omega\iota)+nk_n+2n(n+1)\nu\alpha/a^2\} \\ +B\{(2\omega\iota\alpha+nk_n+2n\nu\alpha/a^2)J_n(ha)-2n\nu\alpha/a^2\,.\,haJ_n'(ha)\} \\ +B'\{(2\omega\iota\alpha+nk_n+2n\nu\alpha/a^2)Y_n(ha)-2n\nu\alpha/a^2\,.\,haY_n'(ha)\} \end{array}\right\} = 0\ (70).$$

The elimination of the four constants A, A', B, B' from equations (66), (67), (69), (70) leads to a somewhat complicated equation to determine the admissible values of α or h which after several reductions can be put into the form

$$\left.\begin{array}{llll} \ldots+2\omega\iota)+nk_n-4n(n-1)\nu\alpha/a^2, & 2n(n-1), & b^{2n}, & 0 \\ \ldots\alpha+2\omega\iota)+nk_n, & 2n(n+1), & a^{2n}, & 2na^n/b^n \\ \ldots-1)a^2J_{n+1}(ha)/ha, & 2haJ_{n+1}(ha)-h^2a^2J_n(ha), & a^nb^nJ_n(hb)-b^{2n}J_n(ha), & hbJ_{n+1}(hb) \\ \ldots-1)a^2Y_{n+1}(ha)/ha, & 2haY_{n+1}(ha)-h^2a^2Y_n(ha), & a^nb^nY_n(hb)-b^{2n}Y_n(ha), & hbY_{n+1}(hb) \end{array}\right| = 0 \ldots\ldots (71).$$

By putting $b = 0$ in this equation it reduces, as it should, to the form (33).

Unfortunately, however, this equation does not admit of having its roots discussed with the same facility as does (35). It cannot *in general* have real or conjugate complex roots unless $\omega = 0$, since by the results (ii), (iii) of § 10 this is impossible when $b = 0$, moreover the general statements of § 11 shew that a purely imaginary root is impossible.

The condition that the real and imaginary parts of one of the roots of (71) may vanish and change sign is that $k_n = 0$, k_n being given by (68). In order that the system may be secularly stable, k_n must be positive for all values of n except zero.

This must be true even if $n = 1$, for unlike (33) the equation (71) does not then split into factors of which the roots are essentially positive. The condition of secular stability is therefore

$$2\pi\gamma(\sigma - \rho) b^2/a^2 - \omega^2 > 0 \quad \text{.............. (72)},$$

which when $\omega^2 = 0$ leads to the condition

$$\sigma > \rho \quad \text{............................ (73)}.$$

It is well known that unless this is fulfilled the liquid will all collect on one side of the solid cylinder.

(8) *The Effect of Surface Tension on Chemical Action.* By J. J. THOMSON, M.A., F.R.S., Cavendish Professor, and J. MONCKMAN (D.Sc. Lond.), Downing College.

IN a paper on "Some Applications of Dynamics," *Phil. Trans.* 1887, Part I., see also *Applications of Dynamics to Physics and Chemistry*, p. 234, one of us has shown that when a number of dilute solutions of reagents are mixed and allowed to settle into equilibrium, then if κ be the coefficient of the chemical action, *i.e.* the value in the state of equilibrium of

$$\frac{x^d y^e z^f \dots}{\xi^a \eta^b \zeta^c \dots},$$

where x, y, z are the masses of the substances which appear as the chemical action goes on, ξ, η, ζ the masses of those which disappear, and a, b, c... the electrochemical equivalents of the substances divided by their molecular weights: the value of κ is altered by the action of the surface tension and the alteration is expressed by the equation

$$\frac{\delta\kappa}{\kappa} = \frac{\mathfrak{c}\, 273}{a \times 1.1 \times 10^{11}} \frac{d}{\theta\, d\xi}(ST),$$

where $\mathfrak{c}$ is the molecular weight of the substance, whose mass is ξ, S the area of the surface exposed, T the surface tension and θ the absolute temperature. Thus if ST increases as ξ increases $\delta\kappa$ is positive, that is ξ, η, ζ are diminished, or in other words the action of the surface tension tends to stop that chemical change which is accompanied by an increase in the value of ST. This is the

effect on the ultimate state of equilibrium, but if the surface tension affects that state it will also affect the rate at which the system approaches it; this is much more easily measured, and the following experiments were undertaken to see if any effect of the kind exists. The reactions on which the effects of surface tension were tried were the following.

1. A solution of sugar acting upon one of permanganate of potash destroys the colour, and gives a reddish precipitate.

2. A solution of hyposulphite of soda acting upon one of bichromate of potash changes its colour, and gives a yellowish green precipitate.

3. Peroxide of hydrogen bleaches aniline very slowly.

4. It has the same effect upon eosine.

5. And upon infusion of roses.

6. Dilute nitric acid bleaches indigo solution.

Method.—At first we attempted to form films that would hold the necessary salts in solution, but without success. Next capillary tubes were tried, and lastly placing the liquid between sheets of glass.

Moderately dilute solutions were mixed in a test-tube and surrounded with black velvet to exclude the light. By means of a pipette a little of the liquid was withdrawn and six drops placed in the centre of a sheet of perfectly clean and dry glass, over this another sheet was laid and the whole covered with black velvet.

After a time, another equal quantity was withdrawn from the test-tube, and placed between sheets of glass as before, and the colour compared with the first. This process was continued until a very decided difference appeared between the first plates and the last.

The surface tension was determined from time to time by measuring the height the liquid rose in a capillary tube.

The changes in the density of the liquids were very small, less than one in four hundred, and not always of a nature to assist the capillary action, so that it is impossible to explain the results obtained by that means.

The experiments may be tabulated thus:

Names of bodies mixed.	Changes in surface tension.	Effect of placing the liquid in thin films between glass plates upon the chemical action.
1. Solutions of Sugar and of Permanganate of Potash	decreased,	quickened.
2. Hyposulphite of Soda and Bichromate of Potash	increased,	retarded.
3. Peroxide of Hydrogen and Aniline (Blue)	increased,	retarded.
4. Another specimen of Peroxide of Hydrogen and three other Aniline Colours (Red, Mauve and Dark Blue)	decreased,	quickened.
5. Peroxide of Hydrogen and Eosine	increased,	retarded.
6. The same and infusion of Roses	decreased,	quickened.
7. Indigo and Nitric Acid	decreased,	quickened.
8. Chloral and Potassic Hydrate	increased,	diminished. See Liebreich, *Phil. Mag.* v. 33, p. 468.

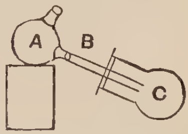
In experiment 7 there was at first an increase in the length of the capillary column, produced by evolved gas, but after standing twelve hours this disappeared and there was a decrease.

Thus when the surface tension of the surface separating the solution from air increases as chemical action goes on, the action is retarded and *vice versâ*.

There is one point which requires explanation in the above results; it is evident from them that though the films are in contact with glass yet it is the surface tension of the surface in contact with air which determines the alteration in the rate of chemical action, for since the surface tension against glass diminishes when that of the surface in contact with air increases and *vice versâ*, if the phenomena had been governed by the surface tension of the glass liquid surface they would have gone in the opposite way. We can we think see the reason for this if we consider how the energy which shows itself as surface tension arises. Let us consider a thin film near the surface, the difference between the energy in this film and that in one of the same thickness in the body of the fluid arises partly from the unbalanced attraction on it of the rest of the liquid and partly from that of the substances against which it is in contact. The

influence of the latter will probably extend to a greater depth than that of the former, so that though a very thin layer immediately in contact with the glass is influenced by its attraction there will be a layer beyond this which is governed by the attraction of the liquid and whose energy changes in the same way as the surface tension of the surface separating it from air, the fluid in this layer can move about more freely than that next the glass, and thus new portions of fluid are continually coming under the same influence, so that its effect on the *rate of change* gets multiplied and thus gains the upper hand over the effect produced by the glass.

In *Applications of Dynamics to Physics and Chemistry*, p. 191, it is shown that when a solution of a salt flows through a tube, the solution will get stronger as it flows if the surface tension of the surface separating the solution from air diminishes as the strength of the solution increases and *vice versâ*. The following experiments were made to illustrate this point.



The apparatus consisted of a two-necked globe A, a glass tube B, and a receiver C.

In B a small piece of cotton-wool was first introduced and afterwards enough fine silica to fill half-an-inch of the tube. These were well compressed and another equal quantity of silica introduced, and so on until nearly full, when a little cotton-wool was pressed into the tube to prevent the powder being carried away by the liquid. So closely was the silica pressed together that although B was not more than 8 inches in length only 1 c.c. of water passed through in 3 days.

Three solutions whose surface tension were greater than water were first tried. These were sulphate of copper, chloride of iron and permanganate of potash, and in all cases the salt was retained behind the water.

To test the matter more fully two liquids were mixed with the water whose surface tension is less than that of water. A little paraffin was shaken up with hot water, by which means a very small quantity was dissolved. After being in the apparatus about a week the liquid in A and that portion which had passed through into C were examined, and it was found that the surface tension in C was less; at the end of the second week the same appeared.

The hydrochloric was diluted with an equal volume of water and placed in a similar bottle. This gave the same result as the others.

Now, the surface tension of solutions of salts is, in almost all

cases, greater than that of pure water, consequently the removal of the salts will decrease the tension.

The rain falls upon the surface of the earth, part of it flows into the rivers, but another portion sinks into the ground and passing through the soil sinks into any porous beds that may lie underneath. In some formations sandstones play an important part, not only in constituting a considerable portion of the thickness, but also in giving character to the hills and valleys in which they occur; their presence can often be traced for miles where no single exposure of rock occurs in the land by signs known to all experienced geologists; among others by the occurrence of springs at the foot of an escarpment. These are produced by the rain-water which has entered them probably at a considerable distance and slowly forced its way through the rock or through cracks until it reaches the spring.

The action of these rocks as capillary tubes removing the saline matter in solution is one cause of the excellence of the water supply in districts where sandstones prevail.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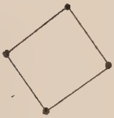
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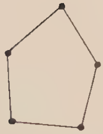
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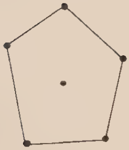
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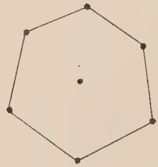
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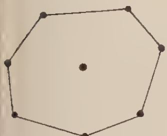
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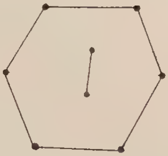
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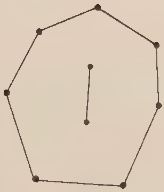
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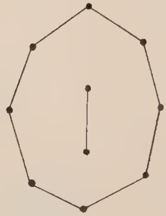
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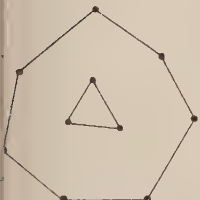
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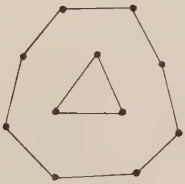
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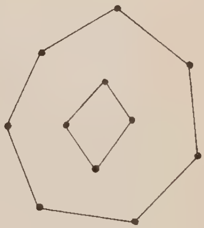
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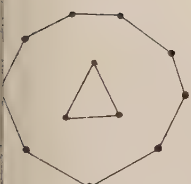
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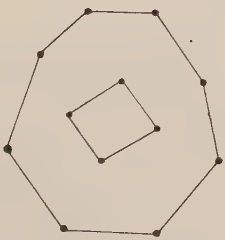
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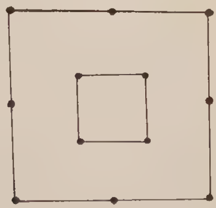
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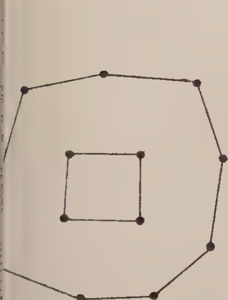
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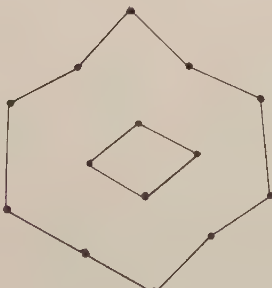
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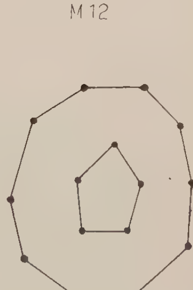
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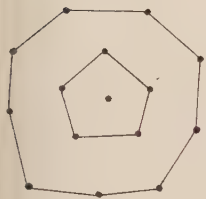

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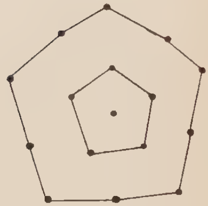

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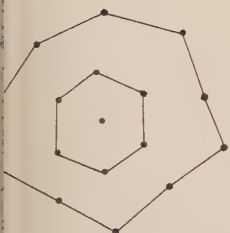

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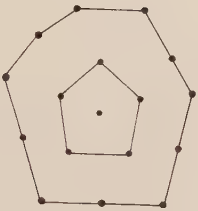

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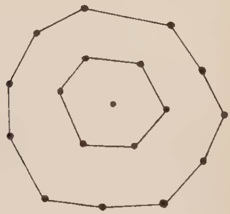

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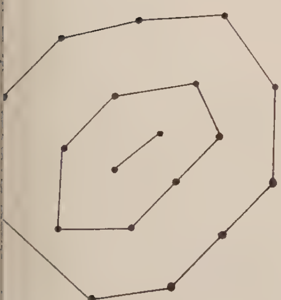

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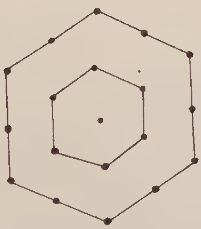

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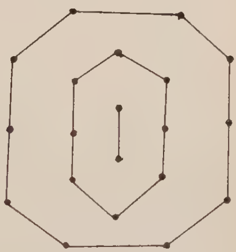

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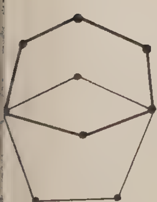

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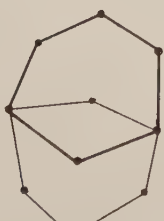

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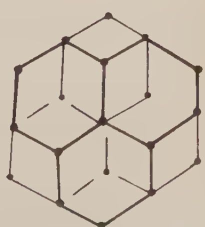

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 29, 1888.

ANNUAL GENERAL MEETING.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

THE following Officers and Members of Council were elected for the ensuing year:

President:

Mr J. W. Clark.

Vice-Presidents:

Prof. Stokes, Prof. Cayley, Dr Routh.

Treasurer:

Mr Glazebrook.

Secretaries:

Mr Larmor, Mr Pattison-Muir, Mr Harmer.

New Members of Council:

Prof. Hughes, Dr Glaisher, Mr Shaw, Mr Gardiner, Mr Bateson.

The names of the benefactors of the Society were recited by the Secretary.

The following communications were made to the Society:

(1) *The Motion of a Solid in a Liquid when the impulse reduces to a couple.* By A. E. H. LOVE, B.A., St John's College.

1. AMONG the possible modes of steady motion of a solid in a liquid, two are of special interest. It has been shown by Kirchhoff that any solid is capable of a steady motion of translation parallel to either of three mutually perpendicular directions, and we owe to Lamb the discovery that when the force-resultant of the impulse vanishes there are three possible steady motions, in which the solid moves on one or other of three screws fixed in it. These screws are at right angles to each other and do not in general intersect, but lie along three alternate edges of a rectangular parallelepiped. Lamb also pointed out that, in case the impulse reduces to a couple, the motion whether steady or not is given by equations which are completely integrable, and described a geometrical representation of it. The object of this paper is to express the motion and position of the solid in this case explicitly in terms of the time. It seems well in the first place briefly to recapitulate Lamb's theory.

2. Let us suppose then that the axes of reference fixed in the solid are parallel to the three directions of steady translation. The kinetic energy in the general case being T, where

$$2T = (c_{11}, c_{22}, c_{33}, c_{44}, c_{55}, c_{66}, c_{12}, \ldots c_{14}, \ldots \between u, v, w, p, q, r)^2,$$

the choice of axes reduces the part

$$(c_{11}, c_{22}, c_{33}, c_{23}, c_{31}, c_{12} \between u, v, w)^2$$

to a sum of squares, and the three axes of steady translation are parallel to those of the quadric

$$(c_{11}, \ldots c_{23}, \ldots \between x, y, z)^2 = 1.$$

Supposing these axes to be the axes of reference, c_{23}, c_{31}, and c_{12} vanish, and the three conditions expressing that the impulse reduces to a couple give

$$c_{11}u = -(c_{14}p + c_{15}q + c_{16}r),$$

and two similar equations, and the kinetic energy is thus reduced to a quadratic function T' of p, q, r, viz.:—

$$2T' = p^2\left(c_{44} - \frac{c_{14}^2}{c_{11}} - \frac{c_{24}^2}{c_{22}} - \frac{c_{34}^2}{c_{33}}\right) + \ldots$$

$$+ 2qr\left(c_{56} - \frac{c_{15}c_{16}}{c_{11}} - \frac{c_{25}c_{26}}{c_{22}} - \frac{c_{35}c_{36}}{c_{33}}\right) + \ldots,$$

and the axes of steady motion are parallel to those of the quadric

$$T'(x, y, z) = \text{const.}$$

We may now show how by a choice of origin u, v, w can become the partial differential coefficients with respect to p, q, r of a quadratic function of p, q, r. If we take any point (x, y, z) as origin, then in $2T$ we must write

$$u + ry - qz, \quad v + pz - rx, \quad w + qx - py$$

instead of u, v, w, and the coefficients in the new energy function will be found by writing

$$c_{26} - c_{22}x, \quad c_{35} + c_{33}x, \quad \text{for } c_{26}, \; c_{35}$$

and similar expressions with y, and z for the other coefficients of the same sort. If then

$$2x = \frac{c_{26}}{c_{22}} - \frac{c_{35}}{c_{33}}, \quad 2y = \frac{c_{34}}{c_{33}} - \frac{c_{16}}{c_{11}}, \quad 2z = \frac{c_{15}}{c_{11}} - \frac{c_{24}}{c_{22}},$$

we shall have $\dfrac{c_{26}'}{c_{22}} = \dfrac{c_{35}'}{c_{33}} = \alpha$ say, where c_{26}' is the new c_{26}, and similar expressions for the other new coefficients. In this case

$$u = -\frac{\partial\Theta}{\partial p}, \quad v = -\frac{\partial\Theta}{\partial q}, \quad w = -\frac{\partial\Theta}{\partial r},$$

where $$2\Theta = \left(\frac{c_{14}}{c_{11}}, \frac{c_{25}}{c_{22}}, \frac{c_{36}}{c_{33}}, \alpha, \beta, \gamma \middle)\!\middle(p, q, r\right)^2.$$

The point here chosen as origin is the centre of the parallelepiped whose three alternate edges are the possible axes of steady motion with no force-resultant of impulse.

Lamb's construction for the motion is then as follows:—Construct an ellipsoid $T'(x, y, z) = \text{const.}$ whose centre is at the point above determined, and let it roll on a plane which is parallel to the plane of the resultant impulsive couple (in this case the whole impulse of the motion), with an angular velocity proportional to the radius vector of the point of contact; if the ellipsoid start with its principal axes parallel to the axes of the possible steady motions, they will always remain so. Next draw the quadric $\Theta(x, y, z) = \text{const.}$, and let the instantaneous axis OI about which the solid is rotating, cut this quadric in P, and draw the central perpendicular OM on the tangent plane at P; then the whole system is to be moved forward with linear velocity which varies inversely as the product $OP \,.\, OM$, and whose direction at any instant is parallel to MO.

3. Now it will be most convenient to take as axes of reference three axes fixed in the solid parallel to those of the steady motions when there is no force-resultant of the impulse. We shall call these for shortness the three *principal rotational axes* of the solid, and the ellipsoid $T'(x, y, z) = \text{constant}$ when referred to these, its principal axes, we shall call the *rotational ellipsoid*. The ellipsoid $\Theta(x, y, z) = \text{constant}$ when referred to the same axes will be called the *translational ellipsoid*, since the former serves to give the motion of rotation of the solid, and the latter that of translation. The principal axes of the translational ellipsoid do not in general coincide with those of the rotational ellipsoid, so that we must take for the equation of the former

$$(a, b, c, f, g, h \between x, y, z)^2 = -\delta \text{ ....(1)},$$

and for that of the latter

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = \epsilon \text{.......................(2)}.$$

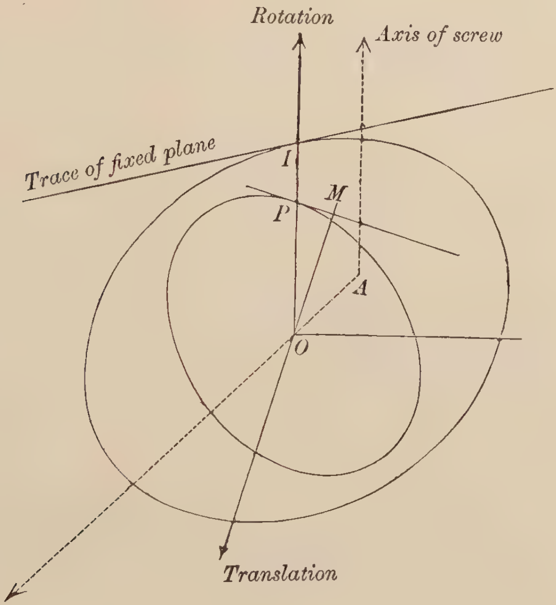
Let G be the constant impulsive couple, and $\frac{1}{2}E$ the constant energy of the motion. The ellipsoid (2) is to be made to roll on a plane, which is normal to the axis of the impulsive couple G; the distance of the centre of the ellipsoid from the plane is to be $\varpi = \frac{\sqrt{\epsilon E}}{G}$, and the angular velocity about the radius vector OI to the point of contact I is $\sqrt{\frac{E}{\epsilon}}\,\rho$, where ρ is the length of OI. It is to be noted that

$$\left.\begin{aligned} E &= p^2/A^2 + q^2/B^2 + r^2/C^2, \\ G^2 &= p^2/A^4 + q^2/B^4 + r^2/C^4 \end{aligned}\right\} \text{.....................(3)}.$$

The quadric (1) is cut by OI in the point P and OM is the perpendicular on the tangent plane at P; calling OP and OM, ρ' and ϖ', the velocity to be impressed on the system of ellipsoid and plane is $\sqrt{\frac{E}{\epsilon}} \cdot \frac{\delta \cdot \rho}{\rho' \cdot \varpi'}$, in the opposite direction to OM. This is Lamb's construction.

4. The solid is twisting on an instantaneous screw whose axis is parallel to OI, and whose pitch is $\frac{\delta}{\rho'^2}$. The axis of the screw cuts a line through O perpendicular to the plane MOP at a distance from O equal to $\frac{\delta \cdot \sqrt{\rho'^2 - \varpi'^2}}{\varpi' \cdot \rho'^2}$, the line is to be drawn in

that direction about which a right-handed wrench would turn OI towards OM. The twist-velocity is $\sqrt{\frac{E}{\epsilon}} \cdot \rho$.



In the figure the plane through the axis of rotation OI, and the direction of translation OM is the plane of the paper, the outer ellipse is the trace on this plane of the rotational ellipsoid, the inner ellipse is that of the translational ellipsoid, the dotted line drawn through A parallel to OI is the axis of the instantaneous screw, and the distance OA is $\frac{\delta}{OP^2} \cdot \frac{PM}{OM}$.

5. We shall now take x, y, z for the coordinates of I referred to the principal axes of the rotational ellipsoid, and l_1, m_1, n_1 for the direction-cosines of OI, also the direction-cosines of the normal at I will be denoted by λ, μ, ν. The direction (λ, μ, ν) is fixed in space.

With this notation, we have

$$x = l_1\rho \quad , \quad y = m_1\rho \quad , \quad z = n_1\rho \quad \text{.........}(4),$$

$$\lambda = \varpi x/\epsilon A^2 \ , \quad \mu = \varpi y/\epsilon B^2 \ , \quad \nu = \varpi z/\epsilon C^2 \quad \text{......}(5),$$

whence $l_1 = \epsilon\lambda A^2/\varpi\rho, \quad m_1 = \epsilon\lambda B^2/\varpi\rho, \quad n_1 = \epsilon\lambda C^2/\varpi\rho \text{......}(6),$

and ρ' is given by the equation

$$(a, b, c, f, g, h \between l_1, m_1, n_1)^2 = -\delta/\rho'^2 \quad \text{.................(7).}$$

The direction-cosines of OM, (ϖ'), are proportional to

$$al_1 + hm_1 + gn_1, \quad hl_1 + bm_1 + fn_1, \quad gl_1 + fm_1 + cn_1,$$

and therefore the direction-cosines of the line MO along which the translation takes place are

$$-\epsilon\,(aA^2\lambda + hB^2\mu + gC^2\nu)/\varpi\rho U,$$

and two similar expressions in which

$$U^2 = \epsilon^2\,[(aA^2\lambda + hB^2\mu + gC^2\nu)^2 + (hA^2\lambda + gB^2\mu + fC^2\nu)^2 + (gA^2\lambda + fB^2\mu + cC^2\nu)^2]/\varpi^2\rho^2 \ldots (8).$$

The velocity of translation V is given by

$$V = \sqrt{\frac{E}{\epsilon}}\,\frac{\delta\rho}{\rho'\varpi'} = \frac{\delta\rho}{n\rho'\varpi'} = \frac{\delta\rho}{n\rho'^2}\,\frac{\rho'}{\varpi'}\,.$$

where $n^2 = \epsilon/E$.

Now $\dfrac{\varpi'}{\rho'} = \cos MOP$

$$= [l_1\,(al_1 + hm_1 + gn_1) + m_1\,(hl_1 + bm_1 + fn_1) + n_1\,(gl_1 + fm_1 + cn_1)]/U$$

$$= -\delta/\rho'^2 U.$$

Hence $V = -\dfrac{\rho}{n}\,U$ ....(9).

It follows that, if u_1, v_1, w_1 denote the components of V along the principal axes of the rotational ellipsoid, we shall have

$$u_1 = \frac{\rho}{n}\,(al_1 + hm_1 + gn_1),$$

with similar expressions for v_1 and w_1, or

$$\left.\begin{aligned} u_1 &= \frac{\epsilon}{n\varpi}\,[aA^2\lambda + hB^2\mu + gC^2\nu] \\ v_1 &= \frac{\epsilon}{n\varpi}\,[hA^2\lambda + bB^2\mu + fC^2\nu] \\ w_1 &= \frac{\epsilon}{n\varpi}\,[gA^2\lambda + fB^2\mu + cC^2\nu] \end{aligned}\right\} \quad \text{...........(10),}$$

in which only the parts in [ ] are functions of the time.

The angular velocity (p, q, r) is given by

$$p : q : r = x : y : z,$$

and taking $x = np, \quad y = nq, \quad z = nr,$

we have

$$n^2\{p^2/A^2 + q^2/B^2 + r^2/C^2\} = \{x^2/A^2 + y^2/B^2 + z^2/C^2\},$$

so that $n^2 = \epsilon/E,$

and $p = \epsilon A^2\lambda/n\varpi, \ q = \epsilon B^2\mu/n\varpi, \ r = \epsilon C^2\nu/n\varpi$.........(11).

6. To determine λ, μ, ν as functions of the time we have only to express that this direction is fixed in space. This gives

$$\left.\begin{array}{l} \dot\lambda + \mu r - \nu q = 0 \\ \dot\mu + \nu p - \lambda r = 0 \\ \dot\nu + \lambda q - \mu p = 0 \end{array}\right\} \text{......................(12).}$$

If M be a factor of homogeneity, and we write

$$\left.\begin{array}{l} \epsilon A^2 - \varpi^2 = M\varpi P \\ \epsilon B^2 - \varpi^2 = M\varpi Q \\ \epsilon C^2 - \varpi^2 = M\varpi R \end{array}\right\} \text{..................(13),}$$

$$\frac{du}{dt} = \frac{M}{n} \text{..............................(14),}$$

the first of equations (12) becomes

$$\frac{d\lambda}{dt} = \frac{\epsilon}{n\varpi}(B^2 - C^2)\,\mu\nu,$$

or

$$\frac{M}{n}\frac{d\lambda}{du} = \frac{M\varpi}{n\varpi}(Q - R)\,\mu\nu,$$

so that these equations are

$$\left.\begin{array}{l} \dfrac{d\lambda}{du} = (Q - R)\,\mu\nu \\ \dfrac{d\mu}{du} = (R - P)\,\nu\lambda \\ \dfrac{d\nu}{du} = (P - Q)\,\lambda\mu \end{array}\right\} \text{........................(15).}$$

7. The integration of this system of differential equations has been considered by Halphen in his Memoir "Sur le mouvement d'un solide dans un liquide" *; the following is an outline of his process (artt. 7—9):

* Liouville's "Journal," 1888; see also the author's "Traité des Fonctions Elliptiques," Part II.

Taking elliptic functions of an argument u, the quantity $\wp u - e_\alpha$ is the square of a uniform function of u, e_α being either of the roots e_1, e_2, e_3 which occur in the fundamental equation

$$\wp'^2 u = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3);$$

it follows that the quantity $(\wp u - e_\alpha)/(\wp v' - e_\alpha)$ is a perfect square, λ^2 say, and then, taking $v = v' + \omega_\alpha$, we have

$$\lambda = \frac{\sigma(u - \omega_\alpha)\,\sigma(v - \omega_\alpha)}{\sigma u\,\sigma v} \exp \eta_\alpha (v + u - \omega_\alpha) \quad \text{.........(16).}$$

Now $(\wp v' - \wp u)/(\wp v' - e_\alpha) = 1 - \lambda^2 = (1 - \lambda)(1 + \lambda)$: but this quantity admits of decomposition into two factors, each a uniform function of u in the form $(\lambda' + i\lambda'')(\lambda' - i\lambda'')$, and thus λ, λ', λ'' are the direction-cosines of some line; these factors are

$$\left.\begin{aligned} \lambda' + i\lambda'' &= C_1 \frac{\sigma(u + v - \omega_\alpha)\,\sigma\omega_\alpha}{\sigma u\,\sigma v} \exp \eta_\alpha (v + u - \omega_\alpha) \\ \lambda' - i\lambda'' &= \frac{1}{C_1} \frac{\sigma(u - v + \omega_\alpha)\,\sigma\omega_\alpha}{\sigma u\,\sigma v} \exp \eta_\alpha (v - u - \omega_\alpha) \end{aligned}\right\} \quad \text{...(17),}$$

where C_1 is an undetermined multiplier.

If in the expressions (16), (17) we change ω_α into any other half-period of the functions, we shall obtain in the same way sets of direction-cosines, (μ, μ', μ''), (ν, ν', ν''), and Halphen proves that the lines thus determined are coorthogonal.

The set of lines thus determined is congruent or incongruent with the axes according as the quantity

$$j = (\lambda'\mu'' - \lambda''\mu')/\nu = \pm 1 \quad \text{..................(18),}$$

and Halphen shows that if the periods be so chosen that

$$\omega_\alpha + \omega_\beta + \omega_\gamma = 0,$$

then
$$j = i \exp(\eta_\alpha\omega_\beta - \eta_\beta\omega_\alpha) = (-1)^{k+1} \quad \text{...........(19),}$$

since
$$\eta_\alpha\omega_\beta - \eta_\beta\omega_\alpha = \tfrac{1}{2}(2k + 1)\, i\pi,$$

where k is an integer.

8. The quantity λ so far as it depends upon u is a special case of the simple type of Hermite's doubly periodic function of the second kind, and products of the quantities λ, μ, ν are in like manner doubly periodic functions of the second kind.

Taking the formula for decomposition into simple elements

$$\frac{\sigma(a_1+a_2)}{\sigma a_1 \sigma a_2}\frac{\sigma(u-a_1)\sigma(u-a_2)}{\sigma^2 u}\exp(\zeta a_1+\zeta a_2)u$$
$$=\frac{d}{du}\left[\frac{\sigma(u-a_1-a_2)}{\sigma u}\exp(\zeta a_1+\zeta a_2)u\right]\ldots(20),$$

and in it writing $a_1=\omega_\alpha$, $a_2=\omega_\beta$, we find that there is a differential equation of the form

$$\frac{1}{\lambda\mu}\frac{d\nu}{du}=\text{constant},$$

viz. this equation is

$$\frac{1}{\lambda\mu}\frac{d\nu}{du}=-\exp(\eta_\beta\omega_\alpha-\eta_\alpha\omega_\beta)\frac{\sigma v\,\sigma(v-\omega_\alpha-\omega_\beta)\,\sigma(\omega_\alpha-\omega_\beta)}{\sigma(v-\omega_\alpha)\,\sigma(v-\omega_\beta)\,\sigma\omega_\alpha\,\sigma\omega_\beta},$$

decomposing the right-hand side into simple elements, we find

$$\frac{1}{\lambda\mu}\frac{d\nu}{du}=\ddot{ij}\left[\zeta(v-\omega_\alpha)-\zeta(v-\omega_\beta)+\zeta\omega_\alpha-\zeta\omega_\beta\right].$$
$$=\frac{1}{2}\ddot{ij}\left[\frac{\wp' v}{\wp v-e_\alpha}-\frac{\wp' v}{\wp v-e_\beta}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

It thus appears that the differential equations (15) for λ, μ, ν as functions of u are satisfied by

$$\left.\begin{aligned}\lambda&=\frac{\sigma(u-\omega_\alpha)\,\sigma(v-\omega_\alpha)}{\sigma u\,\sigma v}\exp\quad\eta_\alpha(v+u-\omega_\alpha)\\ \mu&=\frac{\sigma(u-\omega_\beta)\,\sigma(v-\omega_\beta)}{\sigma u\,\sigma v}\exp\quad\eta_\beta(v+u-\omega_\beta)\\ \nu&=\frac{\sigma(u+\omega_\gamma)\,\sigma(v+\omega_\gamma)}{\sigma u\,\sigma v}\exp-\eta_\gamma(v+u+\omega_\gamma)\end{aligned}\right\}\ldots(22),$$

provided

$$P=-\frac{j}{2i}\frac{\wp' v}{\wp v-e_\alpha},\quad Q=-\frac{j}{2i}\frac{\wp' v}{\wp v-e_\beta},\quad R=-\frac{j}{2i}\frac{\wp' v}{\wp v-e_\gamma}\ldots(23).$$

9. From these and equations (13) we can obtain the constants on which the elliptic functions depend, viz. we have three such equations as

$$\frac{(\epsilon B^2-\varpi^2)(\epsilon C^2-\varpi^2)}{\varpi^2}=-M^2\frac{\wp'^2 v}{4(\wp v-e_\beta)(\wp v-e_\gamma)}$$
$$=-M^2(\wp v-e_\alpha)\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

Remembering that $e_\alpha + e_\beta + e_\gamma = 0$, we obtain

$$-3M^2 e_\alpha = \left[\frac{(\epsilon C^2 - \varpi^2)(\epsilon A^2 - \varpi^2)}{\varpi^2} + \frac{(\epsilon A^2 - \varpi^2)(\epsilon B^2 - \varpi^2)}{\varpi^2} - 2\,\frac{(\epsilon B^2 - \varpi^2)(\epsilon C^2 - \varpi^2)}{\varpi^2}\right] \ldots\ldots (25),$$

from which e_β and e_γ can be found by cyclical interchanges of the letters A, B, C.

From (13) and (23), we have

$$\frac{(\epsilon A^2 - \varpi^2)(\epsilon B^2 - \varpi^2)(\epsilon C^2 - \varpi^2)}{\varpi^8} = M^3 \frac{j}{8i} \frac{\wp'^3 v}{(\wp v - e_\alpha)(\wp v - e_\beta)(\wp v - e_\gamma)}$$

$$= M^3 \frac{j}{2i} \wp' v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

Equations (24) and (26) give $\wp v$ and $\wp' v$, so that v is completely determined, to a period *près*.

The formulæ (16), (17), and those which can be derived from them by substituting the half-periods ω_β and ω_γ instead of ω_α give, as shown by Halphen, a complete representation of the motion when the ellipsoid (2) rolls on one of its tangent planes. He further remarks that the discrimination of $e_\alpha \ldots$ is made at once by observing that these are all real and $e_1 > e_2 > e_3$. He shows that u diminished by an odd multiple of the purely imaginary half-period ω_3 is real and is determined without ambiguity, and that the cosines $(\lambda, \mu, \nu) \ldots\ldots$ as determined by the formulæ are real and less than unity. For this he finds the value of C_1 in (17) to be

$$C_1 = C_0 \exp\left(-\frac{ij\varpi}{M} - \zeta v\right) u \ldots\ldots\ldots\ldots\ldots\ldots (27),$$

C_0 being an arbitrary constant. This is done by using the equations

$$\left.\begin{aligned} \dot{\lambda}' + \mu' r - \nu' q &= 0, \\ \text{and} \qquad \dot{\lambda}'' + \mu'' r - \nu'' q &= 0 \end{aligned}\right\}.$$

Hence the orientation of the solid at any time is completely determined by the equations (16) and (17), and by those formed by substituting therein ω_β and ω_γ successively for ω_α.

10. The translation of the solid is represented by impressing upon the system of rolling ellipsoid and plane a velocity whose components parallel to the axes of the rotational ellipsoid are given by (10), the plane remaining parallel to itself.

Now let ξ, η, ζ be the coordinates of the centre of the rotational ellipsoid referred to axes fixed in space, of which ξ is normal to the plane of the resultant impulsive couple, on which the rotational ellipsoid rolls, and η, ζ are parallel to axes fixed in this plane, then we have

$$\left.\begin{aligned}\dot{\xi} &= \lambda\, u_1 + \mu\, v_1 + \nu\, w_1 \\ \dot{\eta} &= \lambda' u_1 + \mu' v_1 + \nu' w_1 \\ \dot{\zeta} &= \lambda'' u_1 + \mu'' v_1 + \nu'' w_1\end{aligned}\right\}\text{..................(28)},$$

where u_1, v_1, w_1 are given by (10). These equations become

$$\frac{d\xi}{du} = \frac{\epsilon}{M\varpi}\Big[aA^2\lambda^2 + bB^2\mu^2 + cC^2\nu^2 + f(B^2+C^2)\,\mu\nu + g\,(C^2+A^2)\,\nu\lambda$$
$$+ h\,(A^2+B^2)\,\lambda\mu\Big]\text{...(29)},$$

$$\frac{d}{du}(\eta + i\zeta) = \frac{\epsilon}{M\varpi}\Big[aA^2\lambda\,(\lambda' + i\lambda'') + bB^2\mu\,(\mu' + i\mu'') + cC^2\nu\,(\nu' + i\nu'')$$
$$+ f\{C^2\nu(\mu' + i\mu'') + B^2\mu\,(\nu' + i\nu'')\} + g\,\{A^2\lambda\,(\nu' + i\nu'')$$
$$+ C^2\nu(\lambda' + i\lambda'')\} + h\,\{B^2\mu\,(\lambda' + i\lambda'') + A^2\lambda\,(\mu' + i\mu'')\}\Big]$$
$$\text{......(30)},$$

and $\dfrac{d}{du}(\eta - i\zeta)$ is obtained by writing $-i$ for i in this.

11. To integrate the equation for ξ we have only to observe that each of the terms on the right is either an ordinary elliptic function, or else a doubly periodic function of the second kind of the same form as that on the left of (20).

Thus $$\lambda^2 = \frac{(\wp u - e_\alpha)\,(\wp v - e_\alpha)}{(e_\alpha - e_\beta)\,(e_\alpha - e_\gamma)}$$
$$= -\frac{\wp v - e_\alpha}{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}\,\frac{d}{du}\,(\zeta u + u\,.\,e_\alpha)\ \text{...............(31)},$$

and similarly for μ^2 and ν^2 by cyclical interchanges of the letters α, β, γ.

Also $\lambda\mu$ is proportional to

$$\frac{\sigma\,(u - \omega_\alpha)\,\sigma\,(u - \omega_\beta)}{\sigma^2 u}\exp\,(\eta_\alpha + \eta_\beta)\,u,$$

which is equal to

$$\frac{\sigma\omega_\alpha\,\sigma\omega_\beta}{\sigma\,(\omega_\alpha + \omega_\beta)}\,\frac{d}{du}\left[\frac{\sigma\,(u - \omega_\alpha - \omega_\beta)}{\sigma u}\exp(\eta_\alpha + \eta_\beta)\,u\right],$$

and $\mu\nu$, $\nu\lambda$ can be prepared for integration in the same way.

Hence

$$\xi - \xi_0 = \frac{\epsilon}{M\varpi}\left[aA^2 \frac{\wp v - e_\alpha}{(e_\gamma - e_\alpha)(e_\alpha - e_\beta)}(\zeta u + u e_\alpha) + \ldots + \ldots \right.$$
$$+ h(A^2 + B^2)\frac{\sigma(v - \omega_\alpha)\,\sigma(v - \omega_\beta)}{\sigma^2 v}\,\frac{\sigma\omega_\alpha\,\sigma\omega_\beta}{\sigma(\omega_\alpha + \omega_\beta)}$$
$$\frac{\sigma(u - \omega_\alpha - \omega_\beta)}{\sigma u}\exp\{(\eta_\alpha + \eta_\beta)(u + v) - \eta_\alpha\omega_\alpha - \eta_\beta\omega_\beta\}$$
$$\left. + \ldots + \ldots \right] \qquad (32),$$

where ξ_0 is a constant of integration and the terms not expressed are to be found from those put down by cyclical interchanges of the letters α, β, γ; A, B, C; a, b, c; f, g, h.

12. The equations for $\eta + i\zeta$ and $\eta - i\zeta$ do not admit of complete integration in this form. If e.g. we attempt to prepare the term $\lambda(\lambda' + i\lambda'')$ for integration, the part depending on u is

$$\frac{\sigma(u - \omega_\alpha)\,\sigma(u + v - \omega_\alpha)}{\sigma^2 u}\exp\left(2\eta_\alpha - \zeta v - \frac{ij\varpi}{M}\right)u,$$

which is equal to

$$\frac{\sigma\omega_\alpha\,\sigma(\omega_\alpha - v)}{\sigma(2\omega_\alpha - v)}\left[\frac{d}{du}\left\{\frac{\sigma(u - 2\omega_\alpha + v)}{\sigma u}\exp\left(2\eta_\alpha - \zeta v - \frac{ij\varpi}{M}\right)u\right\}\right.$$
$$-\left\{2\eta_\alpha - \zeta v - \frac{ij\varpi}{M} - \zeta\omega_\alpha - \zeta(\omega_\alpha - v)\right\}$$
$$\left.\times\frac{\sigma(u - 2\omega_\alpha + v)}{\sigma u}\exp\left(2\eta_\alpha - \zeta v - \frac{ij\varpi}{M}\right)u\right],$$

by the formula for decomposition into simple elements.

And since
$$\zeta(\omega_\alpha - v) = \zeta\omega_\alpha - \zeta v + \frac{1}{2}\frac{\wp' v}{\wp v - e_\alpha}$$
$$= \eta_\alpha - \zeta v - \frac{ij\varpi}{M}\frac{\epsilon A^2 - \varpi^2}{\varpi^2},$$

the second term becomes

$$\frac{ij\varpi}{M}\frac{\epsilon A^2}{\varpi^2}\frac{\sigma(u - 2\omega_\alpha + v)}{\sigma u}\exp\left(2\eta_\alpha - \zeta v - \frac{ij\varpi}{M}\right)u.$$

This is proportional to

$$\frac{\sigma(u + v)}{\sigma u}\exp\left(-\zeta v - \frac{ij\varpi}{M}\right)u,$$

i.e. to a doubly periodic function of the second kind having one simple pole. The integral of this function appears to be unknown. We may remark that as the function to be integrated has one simple pole and is in all parts of the plane at a finite distance similar to a rational fraction, the integrated function will have a logarithmic critical point $u = 0$, and will be a uniform function of u in every region of the plane at a finite distance and not including this point.

13. To sum up.—The angular velocity and orientation of the solid at any time are completely expressed. Each component of the angular velocity is proportional to a doubly periodic function of the second kind of an argument u proportional to the time, and all the constants are well defined. The direction-cosines of three axes fixed in the solid with reference to axes fixed in space are also completely expressed, each of them as a sum of terms of this form. The distance traversed by the solid in a direction parallel to one of the fixed axes of reference, the axis of the resultant impulsive couple, is in like manner completely expressed as a sum of terms of which three are proportional to u, three are proportional to ζu, and three are doubly periodic functions of the second kind. The distances traversed by the solid in two directions at right angles to this are reduced to quadratures. The velocity of the solid in any direction at any time is completely expressed; in the case of a fixed direction it consists of a sum of terms each of them a doubly periodic function of the second kind having one double pole.

(2) *On Prof. Miller's Observations of Supernumerary Rainbows.* By J. Larmor, M.A., St John's College.

The theory of the supernumerary bows which accompany the primary and secondary rainbows has, it is well known, been placed on an exact mathematical basis by Airy*.

A series of observations on narrow jets of water were made soon after by the late Prof. Miller† with the object of comparing the magnitudes involved with their theoretical values. But so far as appears the comparison was never completed, although the most difficult calculation connected with it was supplied by Prof. Stokes. Prof. Miller contented himself with giving tables of his observations and pointing out that the relative positions of the first few diffraction fringes agreed fairly with the indications of theory.

The rule given by Airy to determine the absolute magnitudes of the bands which accompany the principal bow for homogeneous

* *Camb. Phil. Trans.* Vol. vi. p. 79. † *Camb. Phil. Trans.* Vol. vii. p. 277.

light of index μ is as follows. Obtain the equation of the emerging wave-front, which will be of the form

$$\mu z + bx^3 + \ldots = 0$$

in the neighbourhood of the part efficient in the formation of the bow. The dark and bright bands correspond respectively to the values of m which give the maxima and minima values of the expression

$$\left[\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\, dw\right]^2.$$

If m_r denote such a value, the angular separation of the corresponding band from the geometrical bow is χ, where

$$\chi = \left(\frac{\lambda}{4}\right)^{\frac{2}{3}} \left(\frac{b}{\mu}\right)^{\frac{1}{3}} m_r,$$

λ denoting the wave-length of the light.

To make a comparison, it remains therefore only to determine the value of b. In the case of nature, when the refracting drop is a sphere and the incident beam parallel, this is easily accomplished.

For the geometrical caustic is the evolute of the wave-front; and its radius of curvature ρ at the bow is easily found to be given by

$$\rho = -\frac{6b}{\mu} r^3,$$

where r is the radius of curvature of the wave-front, i.e. the distance of the caustic from it measured along the ray. Now to calculate ρ; let ϕ denote the angle of incidence of a ray, ϕ' its angle of refraction; a the radius of the drop, and p the perpendicular from the centre of the drop on the emergent ray whose deviation is D. Let us consider the n^{th} rainbow. We have

$$\rho = p + \frac{d^2 p}{dD^2},$$

where $$p = a \sin\phi,$$

$$D = 2(\phi - \phi') + n(\pi - 2\phi')$$

$$= n\pi + 2\phi - 2(n+1)\phi';$$

and as D is stationary at the bow,

$$\frac{dD}{d\phi} = 0, \text{ so that } (n+1)\frac{d\phi'}{d\phi} = 1.$$

Thus

$$\left.\begin{aligned}\sin\phi &= \mu\sin\phi' \\ (n+1)\cos\phi &= \mu\cos\phi'\end{aligned}\right\},$$

which determine the position of the geometrical bow.

Now

$$\frac{dp}{dD} = a\cos\phi \bigg/ \frac{dD}{d\phi};$$

$$\frac{d^2p}{dD^2} = -\left(a\sin\phi\frac{dD}{d\phi} + a\cos\phi\frac{d^2D}{d\phi^2}\right)\bigg/\left(\frac{dD}{d\phi}\right)^3;$$

and r is very great, so that

$$r = \frac{dp}{dD} = a\cos\phi \bigg/ \frac{dD}{d\phi}.$$

Hence

$$\begin{aligned}\frac{\rho}{r^3} &= \frac{-1}{(a\cos\phi)^2}\frac{d^2D}{d\phi^2} \\ &= \frac{2(n+1)}{(a\cos\phi)^2}\frac{d^2\phi'}{d\phi^2} \\ &= \frac{2(n+1)}{(a\cos\phi)^2}\frac{\mu\sin\phi'(n+1)^{-2}-\sin\phi}{\mu\cos\phi} \\ &= -\frac{2}{a^2}\frac{n(n+2)}{(n+1)^2}\frac{\sin\phi}{\cos^2\phi} \\ &= -\frac{2n^2}{a^2}\left(\frac{n+2}{n+1}\right)^2\frac{\{(n+1)^2-\mu^2\}^{\frac{1}{2}}}{(\mu^2-1)^{\frac{3}{2}}},\ *\end{aligned}$$

which gives the value of b/μ.

The observations of Prof. Miller relate to the cases $n=1$, $n=2$, the primary and secondary bows.

Although the formulæ here given apply strictly only to the bands whose angular deviation from the geometrical bow is not considerable, it has been thought well to make the comparison with observation through a considerable range of angle.

I owe to Prof. Stokes the remark that, at a sufficiently great angular distance from the principal bow, the interval between successive bands may be calculated simply from the interference of the two effective rays, as in Young's original *aperçu*.

We first examine the primary bows. Applying the formulæ just

* This result was given in a question in the Mathematical Tripos, June 2, 1888 (*Camb. Univ. Exam. Papers*, 1888, p. 560). I find that the same expression is given in the *Comptes Rendus*, May 28, 1888, by M. Boitel. See also *Philosophical Magazine*, Aug. 1888, p. 239.

obtained to Miller's series marked (C) we obtain the following results.

The index from air to water is given as 1·3346; the radius of the cylinder of water is 0·01052 inch. There is considerable uncertainty as to the value of λ which corresponds to this index, inasmuch as the temperature at the time of observation is not given. If we take it to be 12° C., it appears by interpolation from Landolt and Börnstein's Tables that the light corresponds to a place near the *b* lines in the spectrum, and that we may take its wave-length in air to be 5200×10^{-10} metres.

The value of m for the first bright band is 1·0845 (Airy), and the complete system of succeeding values for the other bands has been calculated by Prof. Stokes*.

Calculating by ordinary logarithms, we obtain $\phi = 59° \, 19'{\cdot}03$, $\phi' = 40° \, 7'{\cdot}2$, and for the radius of the geometrical bow

$$4\phi' - 2\phi = 41° \, 50'{\cdot}7,$$

which agrees sufficiently with Miller's value 41° 50′·4.

The deviation of the first bright band (the primary bow) from its geometrical position comes out from these data to be

$$\chi = -27'{\cdot}8.$$

This series (C) is the most consistent of those given. It consists of seven sets, of which the first two extend to 28 bars. But after the 23rd bar the law of succession breaks down completely, as is confirmed for instance by the fact that observations of the 25th are entirely absent. This is conceivably owing to mixture with another series of bands due to some other caustic, which there destroys the continuity of the system under consideration.

If we exclude the bright primary bow, whose position of maximum was, it appears, difficult to fix upon, the series of 23 dark bands agree very perfectly in the different sets, and correspond very closely throughout their whole range to the theoretical values assigned by Prof. Stokes' table.

The value for the deviation of the primary bow from its geometrical position which best suits the observations is, however, 26′·4, though this is 3 or 4 minutes greater than the observations of the primary alone would give. Calculating from this value, the following series of numbers shows how closely the observed deviations of the first 23 dark bars from the position of the geometrical bow agree with the theory. The observations are the mean of Miller's first three series, and correspond very nearly to the second series.

* *Camb. Phil. Trans.* Vol. IX. part 1. *Collected Papers*, Vol. II. p. 349.

Number of Bar	Deviation Calculated	Deviation Observed
1	1° 0′·8	1° 0′·7
2	1 46·5	1 46·7
3	2 23·6	2 23·7
4	2 57·5	2 58·7
5	3 27	3 28
6	3 55	3 58
7	4 21	4 23·7
8	4 46·5	4 48
9	5 11	5 12·7
10	5 34	5 36·7
11	5 56·5	5 58·7
12	6 18·3	6 20·7

Number of Bar	Deviation Calculated	Deviation Observed
13	6° 39′	6° 42′·7
14	7 0	7 2·7
15	7 21	7 22·7
16	7 40·3	7 41·7
17	7 59	8 0·7
18	8 18	8 18·7
19	8 36·5	8 36·7
20	8 54·6	8 55
21	9 12	9 12
22	9 30	9 29
23	9 48	9 47

The uncertainty in the value adopted by interpolation for the wave-length corresponding to the given index involves an error in the calculated deviations which, it has been found, cannot exceed 1/400 of their values.

An increase of ·00012 in the index of refraction of the light in the drop or cylinder, corresponding to a decrease of 1° C. in temperature, will so affect the value of λ as to diminish the calculated deviations by 1/240 of themselves. An increase of ·0001 in this index will diminish the value of θ_1 by 0′·7, and the radius of the geometrical bow by 1′·9.

But, what is more important to notice, an unobserved decrease of 1° C. in the value assumed for the temperature of the prism of water by which the index is determined will produce a decrease in the value of λ estimated from the index, that will have the same effect on the calculated deviations as the corresponding increase of ·00012 in the true index of refraction would have, viz. a diminution of 1/240 of their amount.

In Professor Miller's observations the temperatures are not recorded. He remarks that all the observations were liable to be affected by a sudden shifting of the bars, which was seen occasionally to take place through a small space to the right or left. It is possible that this may be explained as due to temperature variations in the stream of water.

The discrepancy between this value 26′·4 here adopted for the displacement of the primary bow and the value 27′·8 which is the result of the calculations corresponds to a difference of 12° C. of temperature. The observations would therefore agree exactly with theory if we supposed the hollow prism by which the index is determined to have been filled with water from a reservoir at 0° C. The sudden shiftings in the positions of the bars might then be explained (as above) as due to variations of temperature in the filament of water in which the bars are observed.

From the table of deviations which has been calculated for

this series of experiments, the values which apply for any other index and temperature may now be deduced by introducing corrections according to the data just given: while for different values of the radius of the cylinder the deviation is proportional to $(\text{radius})^{-\frac{2}{3}}$.

In series (A) the index was 1·3318 and the radius of the cylinder of water 0·0103 inch. The light was not so homogeneous, and as a consequence the results are not so concordant. But treating them as has been done for (C), they agree very well with theory so far as the first 10 dark bars, on the hypothesis that the displacement of the primary bow is 29′.

The value of this displacement, deduced from that for (C) by applying the corrections given above for the change of index and of radius of the cylinder, is 29′·4 for a temperature 0° C.: it would be 29′ if the temperature of the water cylinder were 3°·3 C.

In series (E) the radius of the cylinder was 0·00675 inch. The index was somewhat doubtful; the value 1·33453 leads to 41° 52′ as the radius of the geometrical bow; the value 1·3348 leads to 41° 46′·9. The second value of the index may be rejected at once, as not in agreement with the results.

The theoretical value of the displacement of the bright primary bow deduced from that in (C) by the necessary corrections is 36′·5 for 0° C. This agrees with the mean value deduced from the observations of the first 8 bars, which is 36′·3; but the succeeding bars deviate from the positions assigned by the theory.

Consider now the circumstances of the secondary bow ($n=2$). In the series (D) which corresponds to (C) for the primary bow, we find $\phi = 71° 47'{\cdot}35$, $\phi' = 45° 22'{\cdot}75$, radius of geometrical bow $= \pi + 2\phi - 6\phi' = 51° 18'{\cdot}2$.

An increase of ·0001 in the index increases this radius by 1′·93.

Thus for the index 1·33464, the radius is 51° 19′·1, which agrees with Miller's result in series (D).

The theoretical displacement of the bright secondary bow from the geometrical position is 49′·61 for a temperature 12° C.

An increase of ·0001 in the index leads to a decrease of 1/220 of itself in the value of the displacement: so that for a temperature 0° C. at the time of observation the displacement would be 47′·4. The alteration is, as before, due to the different value of λ which corresponds to the given index at the altered temperature. This value is in exact agreement with the result deduced from the deviations of the first and second dark bars, and agrees very well with the succeeding ones; though as usual (in accordance with Miller's remark) the number given for the position of maximum brightness of the first band deviates considerably from it. It is to be noticed that the corresponding set of observations (C) of the primary bow required the same temperature correction.

The following table relates to (D), taking the first set of observations.

Number of Bar	Deviation Calculated	Deviation Observed
1	1° 49′	1° 46′
2	3 10·6	3 8
3	4 17·3	4 17
4	5 16·5	5 16
5	6 10·6	6 10
6	7 1	7 0
7	7 48	7 47
8	8 33·5	8 31
9	9 17	9 14
10	9 58	9 56
11	10 38·5	10 33
12	11 17·2	11 13
13	11° 55′	11° 50′
14	12 32·5	12 24
15	13 8	12 59
16	13 44	13 33
17	14 18	14 6
18	14 51·5	14 38
19	15 25	15 10
20	15 58	15 43
21	16 30	16 13
22	17 1	16 41
23	17 33	17 16

The agreement is not so good as in (C), as might be expected from the greater values of the deviation.

Both (C) and (D) seem to show effects of temperance differences in the stream of water as evidenced by nearly constant differences in the readings in parallel columns persisting for a considerable time. Thus in the primary, a variation of 6° C. in the temperature of the cylinder will alter the position of the geometrical bow by 11′·4. The index was observed before and after the experiments; but the error might not thus be revealed, as the index was found by means of still water in a hollow prism which might be filled from another source of more steady temperature.

The series (B), which corresponds to (A) for the primary bow, is not sufficiently consistent to be of much value. The light was not very homogeneous. The observations of the first three dark bars give a displacement of 52′·8 for the principal bow, as compared with 51′·8 given by the theory.

The first dark bars in the series (F), which corresponds to (E) for the primary bow, yield a value 64′ for the displacement of the principal bow, which agrees very well with that given by the theory, viz. 64′·8 calculated from the first and sixth dark bars, and 65′·5 from the first and third.

(3) *On impulsive stress in shafting, and on repeated loading (Wöhler's laws).* By Professor Karl Pearson.

(4) *Application of the Energy Test to the Collapse of a long thin pipe under external pressure.* By G. H. Bryan, B.A., Peterhouse.

The energy criterion of stability lends itself very readily to the treatment of those thin flexible structures of elastic material which collapse through instability of equilibrium rather than

through insufficient strength of their substance. As I pointed out in a former number of the *Proceedings**, thinness and flexibility are necessary qualifications for this kind of collapse.

I shall here discuss (1) Euler's wire under end thrust, a simple illustration of the method, (2) the two dimensional collapse of an infinitely long thin tube under external pressure. The latter was worked out before I was aware that Prof. Greenhill was extending these investigations; the result differs from one given by Prof. Unwin†, owing to the fact that in the latter certain assumptions are made which cannot be regarded as approximate.

(1) Let us first consider a wire of length l under end thrust T, showing that the results agree with Euler's.

Suppose that in the state of equilibrium x is the co-ordinate of any point on the rod measured from one end, and let the rod be displaced in any plane so that the point x receives a lateral displacement z. Then (as in Lord Rayleigh's *Sound*, Vol. I., p. 136) the potential energy of the whole rod due to longitudinal compression, will be *diminished* by the quantity

$$T\int_0^l \left(\frac{ds}{dx}-1\right)dx \text{ or } \tfrac{1}{2}T\int_0^l \left(\frac{dz}{dx}\right)^2 dx,$$

while the potential energy due to bending will be increased by

$$\tfrac{1}{2}\int_0^l \frac{EI}{\rho^2}\,dx \text{ or } \tfrac{1}{2}EI\int_0^l \left(\frac{d^2z}{dx^2}\right)^2 dx.$$

If the rod is in stable equilibrium the potential energy must be increased by the displacement, therefore for all displacements

$$\tfrac{1}{2}EI\int_0^l \left(\frac{d^2z}{dx^2}\right)^2 dx - \tfrac{1}{2}T\int_0^l \left(\frac{dz}{dx}\right)^2 dx > 0 \text{ ....(1).}$$

If the ends are fixed in position, we may, by Fourier's theorem, take

$$z = \Sigma a_n \sin n\pi x/l,$$

and if they are fixed in direction, we must take

$$z = \Sigma a_n \cos n\pi x/l.$$

Either form substituted in (1) leads to the condition

$$\tfrac{1}{2}EI\Sigma a^2{}_n n^4\pi^4/l^4 - \tfrac{1}{2}T\Sigma a^2{}_n n^2\pi^2/l^2 > 0,$$

for all values of the co-ordinates a_n. If we suppose all of these co-ordinates, except one (a_n), vanish, we find

$$T < EIn^2\pi^2/l^2.$$

* "On the Stability of Elastic Systems." *Camb. Phil. Soc.* Vol. VI. Pt. IV. (1888), p. 199.

† *Min. Proc. Inst. C.E.* 1875.

The right-hand side is least when $n = 1$. Hence the critical value of the thrust is

$$T = EI\pi^2/l^2,$$

as on Euler's theory.

(2) When we apply the same principle to the stability of a tube, we must take the same forms for the displacements as those required in treating the vibrations of a ring or cylinder given by Lord Rayleigh in his *Theory of Sound*, Vol. I., § 233. We suppose the cylinder to be infinitely long, of circular section, radius a, and thickness $2h$ small compared with a. Let P be the constant external hydrostatic pressure, T the thrust across any generating line per unit length so that

$$T = Pa.$$

Let the potential energy of bending of the element ds of the circular section per unit length of cylinder be

$$\tfrac{1}{2}\beta ds\,\{\delta\,(1/\rho)\}^2,$$

so that

$$\beta = \tfrac{2}{3}h^3E/(1-\sigma^2),$$

where E is Young's modulus, and σ is Poisson's ratio.

Suppose the system slightly displaced so that the point on the circumference whose cylindrical (polar) co-ordinates were originally (a, θ) is displaced to (r, ϕ) where

$$r = a + \delta r, \quad \phi = \theta + \delta\theta.$$

If e be the extension of the element of arc ds whose original length was $ad\theta$, we have

$$(ad\theta)^2(1+e)^2 = (ds)^2 = (d\delta r)^2 + (a+\delta r)^2(d\theta + d\delta\theta)^2,$$

so that

$$(1+e)^2 = \frac{1}{a^2}\left(\frac{d\delta r}{d\theta}\right)^2 + \left(1 + \frac{\delta r}{a}\right)^2\left(1 + \frac{d\delta\theta}{d\theta}\right)^2.$$

Hence to the second order of small quantities

$$2e + e^2 = \frac{1}{a^2}\left(\frac{d\delta r}{d\theta}\right)^2 + 2\left(\frac{\delta r}{a} + \frac{d\delta\theta}{d\theta}\right) + \left(\frac{\delta r}{a} + \frac{d\delta\theta}{d\theta}\right)^2 + 2\frac{\delta r}{a}\frac{d\delta\theta}{d\theta}.$$

Now the displacement must be one in which the extension of the surface vanishes to the first order of small quantities*, thus

$$\frac{\delta r}{a} + \frac{d\delta\theta}{d\theta} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

while to the second order

$$2e = \frac{1}{a^2}\left(\frac{d\delta r}{d\theta}\right)^2 + 2\frac{\delta r}{a}\frac{d\delta\theta}{d\theta}.$$

* "On the Stability of Elastic Systems," p. 210.

Whatever be the nature of the displacement $\delta r/a$ may always be expanded in the series

$$\delta r/a = \Sigma_{n=1}^{n=\infty} \{A_n \cos n\theta + B_n \sin n\theta\},$$

and from equation (2) the corresponding value of $\delta\theta$ is

$$\delta\theta = \Sigma_{n=1}^{n=\infty} \{-A_n/n \,.\, \sin n\theta + B_n/n \,.\, \cos n\theta\}.$$

The total increase in circumference is

$$= \int_0^{2\pi} ea d\theta$$

$$= \frac{a}{2}\int_0^{2\pi} \left\{\frac{1}{a^2}\left(\frac{d\delta r}{d\theta}\right)^2 + 2\frac{\delta r}{a}\frac{d\delta\theta}{d\theta}\right\} d\theta$$

$$= \tfrac{1}{2}\pi a \Sigma\{(n^2-2)(A_n^2 + B_n^2)\}.$$

Hence the work done *by* the thrust T (per unit length of the cylinder) is

$$= \tfrac{1}{2}\pi a T \Sigma\{(n^2-2)(A_n^2 + B_n^2)\},$$

so that the *increase* of potential energy due to this cause is

$$\delta W_1 = -\tfrac{1}{2}\pi a T \Sigma\{(n^2-2)(A_n^2 + B_n^2)\}.$$

The volume of the deformed cylinder per unit length is

$$= \frac{1}{2}\int r^2 d\phi = \frac{1}{2}\int (a+\delta r)^2 d(\theta + \delta\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\left(1+\frac{\delta r}{a}\right)^2\left(1+\frac{d\delta\theta}{d\theta}\right) d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}\left\{1 + 2\frac{\delta r}{a} + \frac{d\delta\theta}{d\theta} + \left(\frac{\delta r}{a}\right)^2 + 2\frac{\delta r}{a}\frac{d\delta\theta}{d\theta}\right\} d\theta$$

(neglecting small quantities of the *third* order)

$$= \pi a^2 - \tfrac{1}{2}\pi a^2 \Sigma(A_n^2 + B_n^2).$$

Thus the *increase* of volume is

$$-\tfrac{1}{2}\pi a^2 \Sigma(A_n^2 + B_n^2),$$

and the *increase* of potential energy due to work done against the external pressure is

$$\delta W_2 = -\tfrac{1}{2}\pi a^2 P \Sigma(A_n^2 + B_n^2).$$

Lastly, the increase of potential energy due to bending has been shown by Lord Rayleigh to be

$$\delta V = \pi\frac{\beta}{2a}\Sigma(n^2-1)^2(A_n^2 + B_n^2).$$

The circular form will be stable provided that the potential energy is a true minimum; for this to be the case the total increase of potential energy must be positive for any displacement, so that

$$\delta V + \delta W_1 + \delta W_2 > 0.$$

Hence

$$\pi \frac{\beta}{2a} \Sigma (n^2 - 1)^2 (A_n^2 + B_n^2) - \tfrac{1}{2}\pi a^2 P \Sigma (A_n^2 + B_n^2)$$
$$- \tfrac{1}{2}\pi a T \Sigma (n^2 - 2)(A_n^2 + B_n^2) > 0,$$

whence substituting the value Pa for T, we find

$$\Sigma \{\beta/a^3 . (n^2 - 1)^2 - P(n^2 - 1)\} (A_n^2 + B_n^2) > 0.$$

If the displacement be such that all the A's and B's vanish with the exception of A_n, B_n, the condition of stability will be

$$P < (n^2 - 1)\beta/a^3,$$

or

$$P < \tfrac{2}{3}(n^2 - 1)\frac{h^3}{a^3}\frac{E}{1-\sigma^2}.$$

If P is greater, then the cylinder will become unstable and will collapse into segments, the number of curved arcs being $2n$.

Unwin's formula gives in our notation for the collapsing pressure

$$P = \tfrac{2}{3} n^2 E \frac{h^3}{a^3}.$$

The displacements for which $n = 1$ being only motions of translation of the whole cylinder, the least collapsing pressure corresponds to $n = 2$ and will be

$$P = 2\frac{E}{1-\sigma^2}\frac{h^3}{a^3} = 2\frac{E}{1-\sigma^2}\left(\frac{\text{thickness}}{\text{diameter}}\right)^3.$$

By sufficiently diminishing the ratio of the thickness to the diameter we may make the strains produced in the cylinder by this pressure as small as we please. For if f be the compression of the substance along the circular sections of the cylinder (so that its circumference is diminished in the ratio of $1 - f$ to 1), then

$$\frac{E}{1-\sigma^2} 2hf = T = Pa,$$

whence taking the value of P just found

$$f = \frac{h^2}{a^2} = \left(\frac{\text{thickness}}{\text{diameter}}\right)^2.$$

Therefore f diminishes continually as the ratio of h to a is diminished and by taking the ratio $h : a$ small enough, f may be always made so small that the tube is not strained beyond the limits of elasticity.

Taking the thickness of the tube to be the one-hundredth part of its diameter, and employing the values of E, σ given by Lupton, I find the values of P in C. G. S. atmospheres are for

Glass........................	·88
Brass........................	2·23
Cast iron	2·90
Steel........................	4·73

approximately.

When the tube collapses through instability it does not necessarily follow that it will burst. In the case of "collapse into four segments" here considered the tube begins by becoming elliptical and gets more or less flattened, but unless the bending moment should become too great at any point the tube will not break but will probably pass into a form somewhat resembling

the one here represented, the support afforded by the contact of opposite sides preventing further collapse. Moreover it is important to notice that on this hypothesis *if the pressure be removed, the tube will return to the circular form,* which it would not do if the material gave way at any point.

We may also find the effect of the pressure on the vibrations of the cylinder. If the frequency is $p/2\pi$, then Lord Rayleigh's formula will be replaced by

$$p^2 = \frac{n^2(n^2-1)^2}{n^2+1}\left(\frac{\beta}{\rho a^3} - \frac{P}{(n^2-1)\rho}\right),$$

ρ being the mass per unit area of surface of the tube.

The vibrations will all be lowered in pitch, owing to the variations of the potential energy being diminished. The above formula does not however take into account the sound-waves produced in the medium by which the pressure is transmitted to the surface of the vibrating cylinder.

(5) *On the Expression of Spherical Harmonics of the Second Kind in a Finite Form.* By G. H. BRYAN, B.A., Peterhouse.

1. IN this paper I propose to give a handy and convenient method whereby both Laplace's coefficients of the second kind, and the corresponding associated functions or tesseral harmonics may be exhibited in a finite form. In the case of the former it is hardly necessary to add anything to what has already been done in this direction, but the present method, when applied to the latter functions, will be found much less laborious than the process of differentiation by which they have been hitherto obtained.

Taking as usual

$$Q_n(\mu) = P_n(\mu) \int_\mu^\infty \frac{d\mu}{(\mu^2 - 1)\{P_n(\mu)\}^2},$$

we know that if $\mu > 1$

$$Q_n(\mu) = \tfrac{1}{2} P_n(\mu) \log \frac{\mu + 1}{\mu - 1} - R$$

$$= P_n(\mu) \coth^{-1} \mu - R \quad \text{.....................(1)},$$

where R is a rational integral function of μ of degree $n - 1$.

Also

$$Q_n(\mu) = \frac{n!}{1.3.5\ldots(2n+1)} \left\{ \frac{1}{\mu^{n+1}} + \frac{(n+1)(n+2)}{2.(2n+3)} \frac{1}{\mu^{n+3}} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} \frac{1}{\mu^{n+5}} + \ldots \right\} \quad \text{.......(2)}.$$

In the first form let $P_n(\mu)$ be expressed as an algebraic function of μ, and expand $\coth^{-1}\mu$, that is, $\tanh^{-1}(1/\mu)$ in powers of $1/\mu$. We thus obtain by equating to the second form of $Q_n(\mu)$

$$P_n(\mu)\left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \ldots\right) - R$$

$$\equiv \frac{n!}{1.3.5\ldots(2n+1)} \left\{ \frac{1}{\mu^{n+1}} + \ldots \right\} \quad \text{.........(3)}.$$

Hence the left-hand side cannot contain positive powers of μ, and therefore R must be equal to the terms of positive degree in μ in the expansion of

$$P_n(\mu)\left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \ldots\right).$$

This gives a convenient mode of calculating R, and, hence, of expressing $Q_n(\mu)$ in a finite form.

Examples.—

(1) $$P_3(\mu) = \frac{5\mu^3 - 3\mu}{2},$$

also

$$\frac{5\mu^3 - 3\mu}{2}\left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \dots\right) = \frac{5\mu^2}{2} - \frac{2}{3} + \text{negative powers of } \mu;$$

$$\therefore \quad R = \frac{15\mu^2 - 4}{6}, \text{ and } Q_3(\mu) = \frac{5\mu^3 - 3\mu}{2}\coth^{-1}\mu - \frac{15\mu^2 - 4}{6}.$$

(2) $$P_5(\mu) = \frac{63\mu^5 - 70\mu^3 + 15\mu}{8},$$

also $$\frac{63\mu^5 - 70\mu^3 + 15\mu}{8}\left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \dots\right)$$

$$= \frac{945\mu^4 - 735\mu^2 + 64}{120} + \text{negative powers of } \mu,$$

hence

$$Q_5(\mu) = \frac{63\mu^5 - 70\mu^3 + 15\mu}{8}\coth^{-1}\mu - \frac{945\mu^4 - 735\mu^2 + 64}{120},$$

results which agree with those obtained by the usual methods.

2. The associated function or tesseral harmonic of the second kind of order n, and rank s is proportional to

$$(\mu^2 - 1)^{\frac{1}{2}s} D^s Q_n(\mu),$$

where the symbol D stands for the operator $d/d\mu$.

Now taking $Q_n(\mu)$ in the form (1) and differentiating s times using Leibnitz's theorem, it is evident that the result can be put in the form

$$D^s Q_n(\mu) = \coth^{-1}\mu D^s P_n(\mu) - \frac{R'}{(\mu^2 - 1)^s} \quad \dots\dots\dots\dots\dots (4),$$

where R' is a rational algebraic function involving only positive powers of μ.

Hence $$(\mu^2 - 1)^{\frac{1}{2}s} D^s Q_n(\mu) = \coth^{-1}\mu \,.\, (\mu^2 - 1)^{\frac{1}{2}s} D^s P_n(\mu)$$
$$- R'(\mu^2 - 1)^{-\frac{1}{2}s} \quad \dots\dots\dots\dots\dots\dots\dots\dots (5),$$

and $$(\mu^2 - 1)^s D^s Q_n(\mu) = \coth^{-1}\mu \,.\, (\mu^2 - 1)^s D^s P_n(\mu) - R' \quad \dots .(6).$$

From this formula, R' may be obtained in a similar manner to that which we have exemplified in the case of the zonal harmonics. For taking the form (2) for $Q_n(\mu)$ and differentiating,

the lowest negative power of μ that occurs in $D^s Q_n(\mu)$ will be $\mu^{-(n+s+1)}$. Hence in $(\mu^2-1)^s D^s Q_n(\mu)$ the lowest negative power of μ that occurs is μ^{-n+s-1}, therefore since s is never greater than n, $(\mu^2-1)^s D^s Q_n(\mu)$ only involves negative powers of μ, and hence

$$R' = \text{terms of } (\mu^2-1)^s D^s P_n(\mu) \left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \dots\right),$$

involving positive powers of μ and constant terms.

This determines R', and the corresponding associated function of the second kind is given by equation (5).

Example.—Suppose $n = 4$, $s = 2$, then

$$(\mu^2-1)^2 D^2 P_4(\mu) = \tfrac{15}{2}(\mu^2-1)^2(7\mu^2-1),$$

and
$$\tfrac{15}{2}(\mu^4 - 2\mu^2 + 1)(7\mu^2-1)\left(\frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \dots\right)$$

$$= \tfrac{1}{2}(105\mu^5 - 190\mu^3 + 81\mu);$$

therefore
$$(\mu^2-1)^{2/2} D^2 Q_2(\mu)$$

$$= \tfrac{15}{2}(7\mu^2-1)(\mu^2-1)\coth^{-1}\mu - \frac{105\mu^5 - 190\mu^3 + 81\mu}{2(\mu^2-1)}.$$

3. It is however unnecessary to go through the labour of obtaining the value of $(\mu^2-1)^s D^s P_n(\mu)$ by the method of the last paragraph.

For it is readily proved that

$$(\mu^2-1)^s D^s P_n(\mu), \text{ and } (\mu^2-1)^s D^s Q_n(\mu)$$

are integrals of the equation in z

$$(1-\mu^2)\frac{d^2z}{d\mu^2} + 2(s-1)\mu\frac{dz}{d\mu} + (n+s)(n-s+1)z = 0,$$

hence by solving in series or otherwise

$$(\mu^2-1)^s D^s P_n(\mu) = A\left\{\mu^{n+s} - \frac{(n+s)(n+s-1)}{2\,.\,(2n-1)}\mu^{n+s-2}\right.$$
$$\left. + \frac{(n+s)(n+s-1)(n+s-2)(n+s-3)}{2\,.\,4\,(2n-1)(2n-3)}\mu^{n+s-4} - \dots\right\},$$

where A is some constant, and from the coefficient of μ^{n+s} we find that the proper value is

$$A = \frac{2n(2n-1)\dots(n-s+1)}{2^n n!}.$$

4. From this result the general form of R' can be obtained. For since R' is the portion involving positive powers of μ in the product

$$A\left\{\mu^{n+s}-\frac{(n+s)(n+s-1)}{2.(2n-1)}\mu^{n+s-2}\right.$$
$$\left.+\frac{(n+s)(n+s-1)(n+s-2)(n+s-3)}{2.4.(2n-1)(2n-3)}\mu^{n+s-4}-\dots\right\}$$
$$\left\{\frac{1}{\mu}+\frac{1}{3\mu^3}+\frac{1}{5\mu^5}+\frac{1}{7\mu^7}+\dots\right\};$$

therefore

$$R'=a_1\mu^{n+s-1}+a_2\mu^{n+s-3}+\dots+a_r\mu^{n+s-2r+1}+\dots,$$

where

$$a_r=A\left\{\frac{1}{2r-1}-\frac{(n+s)(n+s-1)}{2.(2n-1)}\frac{1}{2r-3}\right.$$
$$\left.+\frac{(n+s)(n+s-1)(n+s-2)(n+s-3)}{2.4.(2n-1)(2n-3)}\frac{1}{2r-5}-\dots\right\}.$$

5. The principal applications of the functions which we are considering are those relating to potentials of spheroids, and the forms discussed in the present paper are more convenient for calculation in most cases which present themselves, than the well-known expansions in descending powers of μ. In potential problems which relate to oblate spheroids μ is imaginary, but the present methods are equally applicable with the requisite modifications, while the expansions in series become divergent when applied to an oblate spheroid whose eccentricity is greater than $\frac{1}{2}\sqrt{2}$.

For convenience of reference, I have given in the following table the expressions in a finite form of $T_n^s(\mu)$, and $U_n^s(\mu)$ for values of n up to 5 inclusive, $T_n^s(\mu)$ and $U_n^s(\mu)$ being defined as follows:

$$T_n^s(\mu)=(\mu^2-1)^{\frac{1}{2}s}D^sP_n(\mu),$$

$$U_n^s(\mu)=T_n^s(\mu)\int_\mu^\infty\frac{d\mu}{(\mu^2-1)\{T_n^s(\mu)\}^2}=(-1)^s\frac{n-s!}{n+s!}(\mu^2-1)^{\frac{1}{2}s}D^sQ_n(\mu).$$

n	s	$T_n^s(\mu)$	$U_n^s(\mu)$
1	0	$P_1=\mu$	$Q_1=\mu\coth^{-1}\mu-1$
1	1	$(\mu^2-1)^{\frac{1}{2}}$	$-\frac{1}{1\,.\,2}\left\{(\mu^2-1)^{\frac{1}{2}}\coth^{-1}\mu-\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}}\right\}$
2	0	$P_2=\frac{1}{2}(3\mu^2-1)$	$Q_2=\frac{1}{2}(3\mu^2-1)\coth^{-1}\mu-\frac{3}{2}\mu$
2	1	$3\mu(\mu^2-1)^{\frac{1}{2}}$	$-\frac{1}{2\,.\,3}\left\{3\mu(\mu^2-1)^{\frac{1}{2}}\coth^{-1}\mu-\frac{3\mu^2-2}{(\mu^2-1)^{\frac{1}{2}}}\right\}$
2	2	$3(\mu^2-1)$	$+\frac{1}{1\,.\,2\,.\,3\,.\,4}\left\{3(\mu^2-1)\coth^{-1}\mu-\frac{3\mu^2-5\mu}{\mu^2-1}\right\}$
3	0	$\frac{1}{2}(5\mu^3-3\mu)$	$\frac{1}{2}(5\mu^3-3\mu)\coth^{-1}\mu-\frac{15\mu^2-4}{6}$
3	1	$\frac{3}{2}(5\mu^2-1)(\mu^2-1)^{\frac{1}{2}}$	$-\frac{1}{3\,.\,4}\left\{\frac{3}{2}(5\mu^2-1)(\mu^2-1)^{\frac{1}{2}}\coth^{-1}\mu-\frac{15\mu^3-13\mu}{2(\mu^2-1)^{\frac{1}{2}}}\right\}$
3	2	$15\mu(\mu^2-1)$	$\frac{1}{2\,.\,3\,.\,4\,.\,5}\left\{15\mu(\mu^2-1)\coth^{-1}\mu-\frac{15\mu^4-25\mu^2+8}{\mu^2-1}\right\}$
3	3	$15(\mu^2-1)^{\frac{3}{2}}$	$-\frac{1}{1.2.3.4.5.6}\left\{15(\mu^2-1)^{\frac{3}{2}}\coth^{-1}\mu-\frac{15\mu^5-40\mu^3+33\mu}{(\mu^2-1)^{\frac{3}{2}}}\right\}$
4	0	$\frac{1}{8}(35\mu^4-30\mu^2+3)$	$\frac{1}{8}(35\mu^4-30\mu^2+3)\coth^{-1}\mu-\frac{1}{24}(105\mu^3-55\mu)$
4	1	$\frac{5}{2}(7\mu^3-3\mu)(\mu^2-1)^{\frac{1}{2}}$	$-\frac{1}{4\,.\,5}\left\{\frac{5}{2}(7\mu^3-3\mu)(\mu^2-1)^{\frac{1}{2}}\coth^{-1}\mu-\frac{105\mu^4-115\mu^2+16}{6(\mu^2-1)^{\frac{1}{2}}}\right\}$
4	2	$\frac{15}{2}(7\mu^2-1)(\mu^2-1)$	$\frac{1}{3\,.\,4\,.\,5\,.\,6}\left\{\frac{15}{2}(7\mu^2-1)(\mu^2-1)\coth^{-1}\mu-\frac{105\mu^5-190\mu^3+81\mu}{2(\mu^2-1)}\right\}$
4	3	$105\mu(\mu^2-1)^{\frac{3}{2}}$	$-\frac{1}{2.3.4.5.6.7}\left\{105\mu(\mu^2-1)^{\frac{3}{2}}\coth^{-1}\mu-\frac{105\mu^6-280\mu^4+231\mu^2-48}{(\mu^2-1)^{\frac{3}{2}}}\right\}$
4	4	$105(\mu^2-1)^2$	$\frac{1}{1\,.\,2\ldots8}\left\{105(\mu^2-1)^2\coth^{-1}\mu-\frac{105\mu^7-385\mu^5+511\mu^3-279\mu}{(\mu^2-1)^2}\right\}$
5	0	$\frac{1}{8}(63\mu^5-70\mu^3+15\mu)$	$\frac{1}{8}(63\mu^5-70\mu^3+15\mu)\coth^{-1}\mu-\frac{945\mu^4-735\mu^2+64}{120}$
5	1	$\frac{15}{8}(21\mu^4-14\mu^2+1)(\mu^2-1)^{\frac{1}{2}}$	$-\frac{1}{5\,.\,6}\left\{\frac{15}{8}(21\mu^4-14\mu^2+1)(\mu^2-1)^{\frac{1}{2}}\coth^{-1}\mu-\frac{315\mu^5-420\mu^3+113\mu}{8(\mu^2-1)^{\frac{1}{2}}}\right\}$
5	2	$\frac{105}{2}(3\mu^3-\mu)(\mu^2-1)$	$\frac{1}{4\,.\,5\,.\,6\,.\,7}\left\{\frac{105}{2}(3\mu^3-\mu)(\mu^2-1)\coth^{-1}\mu-\frac{315\mu^6-630\mu^4+343\mu^2-32}{2(\mu^2-1)}\right\}$
5	3	$\frac{105}{2}(9\mu^2-1)(\mu^2-1)^{\frac{3}{2}}$	$-\frac{1}{3\,.\,4\ldots.8}\left\{\frac{105}{2}(9\mu^2-1)(\mu^2-1)^{\frac{3}{2}}\coth^{-1}\mu-3\frac{315\mu^7-875\mu^5+753\mu^3-221\mu}{2(\mu^2-1)^{\frac{3}{2}}}\right\}$
5	4	$945\mu(\mu^2-1)^2$	$\frac{1}{2\,.\,3\ldots.9}\left\{945\mu(\mu^2-1)^2\coth^{-1}\mu-3\frac{315\mu^8-1155\mu^6+1533\mu^4-837\mu^2+128}{(\mu^2-1)^2}\right\}$
5	5	$945(\mu^2-1)^{\frac{5}{2}}$	$-\frac{1}{1\,.\,2\ldots.10}\left\{945(\mu^2-1)^{\frac{5}{2}}\coth^{-1}\mu-3\frac{315\mu^9-1470\mu^7+2688\mu^5-2370\mu^3+965\mu}{(\mu^2-1)^{\frac{5}{2}}}\right\}$

(6) *On a machine for describing an Equiangular Spiral.* By HORACE DARWIN, M.A., Trinity College.

The curve is traced by an inked wheel, which is set obliquely on the bar which forms the radius vector, and rolls on the paper.

November 12, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

C. W. C. BARLOW, B.A., Peterhouse, was elected a Fellow.

The following communications were made to the Society:

(1) *Exhibition of photographs of the image formed on the retina by an electric lamp.* By Mr FRIESE-GREEN (communicated by Mr R. T. Glazebrook).

(2) *Suggestion that certain fossils known as* Bilobites *may be regarded as casts of* Balanoglossus. By W. BATESON, M.A., St John's College.

The author stated that he had by chance seen specimens of these fossils, which occur in the Carboniferous series, in the Woodwardian Museum, and had been struck by the resemblance which they bore to *Enteropneusta*. He had subsequently, by the kindness of Professor Hughes, been permitted to examine his large collection of these forms, and had also brought together additional specimens from the place in Westmoreland where Professor Hughes had discovered them.

The likeness which these fossils bear to *Balanoglossus*, in size and surface-markings, is very close, and they are practically a representation in stone of the generative region of that animal. The author suggested therefore that they had been formed as casts of the closely fitting tube of mucus and sand which envelopes parts of the larger species of *Balanoglossus*. He particularly called attention to a specimen which was apparently a cast of the body at the junction of the generative and intestinal regions. The fossils are of considerable length, varying up to about two feet. *Balanoglossus* is the only animal having a comparable structure which reaches such a length. Two species were exhibited of the fossil form which differed from each other very much as *B. salmoneus* (Giard) differs from *B. Robinii* (Giard). The texture of the sandstone was exactly that of the sand at the Glenans Islands where these two forms now occur. In some specimens, as pointed out by Professor Hughes, there were indications that the animals had lived immediately under the surface of the sand, which is not the case with existing forms.

The author wished to express his thanks to Professor Hughes for his assistance in this investigation.

(3) *Notes on a Collection of Spiders, with a list of species taken in the neighbourhood of Cambridge.* By C. WARBURTON, Christ's College.

ALL attempts to preserve spiders in the dry state have hitherto proved ineffectual.

When put up in alcohol, the specimens must either be mounted in some way, and certain specific characteristics concealed, or allowed to lie loosely in tubes, and to present a distorted and unsightly appearance.

For the purposes of exhibition, the former alternative seems preferable, especially if care be taken to minimise as far as possible its disadvantages.

A simple but effective method of mounting specimens is here described, as likely to prove useful to collectors in this and other groups, where no satisfactory dry method of preservation is available.

A specimen tube is filled about one-third full of plaster of Paris powder. Water is added, and the tube corked and shaken, and then laid lengthwise upon a horizontal surface. When the plaster is set, the block is slipped out, smoothed if necessary, and the specimen mounted upon its flat surface with strong gum or "liquid glue"—a substance not dissolved by alcohol.

When replaced, the block of course fits its mould, and cannot crush the specimen, as the width of its flat surface is nearly the diameter of the tube. It moreover affords a white back-ground which is not liable to much discolouration. It is often convenient to mount male and female of a species in the same tube.

The tubes are then labelled and exposed on tiers of shelves, inclined at a small angle to the perpendicular.

Thus arranged, the specimens bear some resemblance to the living species they typify, and present as sightly an appearance as the difficulties of the case will admit.

List of Spiders taken in the neighbourhood of Cambridge.

DYSDERIDES.

DYSDERA

Cambridgii, Thor.,—occasional, on Castle Hill, Gogmagog hills, etc.

HARPACTES

Hombergii, Scop., not rare, at the bottom of Clare wall, and in the court of Christ's College.

OONOPS

pulcher, Temp., rare, on Gogmagog hills.

DRASSIDES.

MICARIA

pulicaria, Sund, frequent, on Castle Hill, Gogmagog hills, etc.

PROSTHESIMA

Petiverii, Scop., rare, Fleam Dyke.
nigrita, Fabr., rare, Fleam Dyke.

Drassus
lapidicoleus, Walck., common.
pubescens, Thor., very rare.

Clubiona
pallidula, Clk., frequent, in ivy leaves.
terrestris, Westr., occasional.
lutescens, Westr., in the fens.
holosericea, De Geer, occasional, in curled leaves.
brevipes, Bl., rare.
comta, C. L. Koch, occasional, in trees.
subtilis, L. Koch, rare, Wicken Fen.

Agröeca
brunnea, Bl., frequent, in grass.

Hecäerge
maculata, Bl., occasional.

DICTYNIDES.

Dictyna
arundinacea, Linn., frequent, on shrubs.
uncinata, Westr., occasional.

Amaurobius
fenestralis, Stroem, common, in dry grass, vegetable débris, etc.
similis, Bl., frequent, in out-houses.
ferox, Walck., frequent.

AGELENIDES.

Tegenaria
Guyonii, Guérin, not rare, in buildings.
Derhamii, Scop., common, in buildings.
campestris, C. L. Koch, frequent, under ledges of walls.

Agelena
labyrinthica, Clk., common, on banks.

Hahnia
elegans, Bl., occasional in Wicken Fen.

Textrix
denticulata, Oliv., rare, enclosure of University Bathing Sheds.

THERIDIIDES.

Theridion
formosum, Clk., rare, Botanical Gardens.
tepidariorum, C. L. Koch, in hot-houses.
pictum, Hahn., frequent, on holly bushes.
sisyphium, Clk., very common, on holly etc.
denticulatum, Walck., occasional, on shrubs etc.
varians, Hahn., common, on bushes.
pulchellum, Walck., rare, on trees.
bimaculatum, Linn., frequent, in grass.
pallens, Bl., not rare, on trees, shrubs etc.

Nesticus
cellulanus, Clk., occasional in damp places, e.g. tanks in Christ's College gardens.

Phyllonethis
lineata, Clk., very common; everywhere.

Steatoda
bipunctata, Linn., not rare, in out-houses.

Neriene
longipalpis, Sund., common, on railings etc.
dentipalpis, Wid., occasional.
rufipes, Sund., taken at the University Bathing Sheds.
rubens, Bl., frequent, in grass.
isabellina, C. L. Koch, rare.

fusca, Bl., occasional.
livida, Bl., occasional.

Walckenäera
Hardii, Bl., very rare; one example taken in the "Backs."
antica, Wid., rare, Wicken Fen.

Pachygnatha
Clerckii, Sund., frequent, in damp grass.
Degeerii, Sund., frequent, in grass.

Linyphia
nebulosa, Sund., in Christ's College garden.
zebrina, Menge, occasional.
leprosa, Ohl., frequent.
tenebricola, Wid., common, in grass.
socialis, Sund., frequent, on trees.
dorsalis, Wid., occasional, in plantations.
bicolor, Bl., frequent, in grass.
bucculenta, Clk., common on Castle Hill, etc.
montana, Clk., frequent, on Clare wall.
triangularis, Clk., common, on bushes.

EPEIRIDES.

Meta
segmentata, Clk., everywhere.

Tetragnatha
extensa, Linn., frequent, on Clare wall etc.

Cyclosa
conica, Pall., rare, in wood on the Gogmagog hills.

Zilla
x-notata, Clk., everywhere.
atrica, C. L. Koch, frequent.

Epeira
cucurbitina, Clk., frequent, on trees, bushes etc.
diademata, Bl., common.
scalaris, Walck., rare, in woods.
arbustorum, C. L. Koch, rare, in woods.
cornuta, Clk., frequent, in nettles etc.
patagiata, Clk., rare, in woods.
sclopetaria, Clk., not rare, on Clare College wall.
umbratica, Clk., not rare, at University Bathing Sheds, etc.

THOMISIDES.

Xysticus
cristatus, Clk., common.
viaticus, C. L. Koch, occasional, on Castle Hill.
pini, Hahn., rare.
lanio, C. L. Koch, frequent in wood on the Gogmagogs.
ulmi, Hahn., rare.
erraticus, Bl., rare.

Oxyptila
atomaria, Panz., Wicken Fen.

Philodromus
aureolus, Clk., common, in fir trees, etc.

Thanatus
hirsutus, Camb. (or striatus, C. L. Koch), frequent in Wicken Fen.

Tibellus
oblongus, Walck., common in grass, Castle Hill etc.

LYCOSIDES.

Ocyale
mirabilis, Clk., occasional, Fleam Dyke etc.

PIRATA
piraticus, Clk., common, near water.

TROCHOSA
ruricola, De Geer, frequent.
terricola, Thor., occasional.

TARENTULA
pulverentula, Clk., common.
audrenivora, Walck., frequent.

LYCOSA
amentata, Clk., very common.
lugubris, Walck., very common but local.
Farrenii, Cambr., rare, in Wicken Fen.
pullata, Clk., frequent.
riparia, C. L. Koch, occasional.
nigriceps, Thor., common.
monticola, C. L. Koch, occasional.

SALTICIDES.

EPIBLEMUM
scenicum, Clk., frequent, on sunny walls.

HELIOPHANUS
cupreus, Walck., rare.

EUOPHRYS
frontalis, Walck, frequent in grass, Castle Hill etc.

ATTUS
pubescens, Fabr., rare, on walls.

(4) *On* Splachnum luteum, Linn. By J. R. VAIZEY, M.A., Peterhouse.

IN the summer of 1887 I went to Norway for the purpose of obtaining a supply of this moss, in order to investigate the structure of the sporophyte; the investigations of Haberlandt[1], Vuillemin[2] and myself[3] in the last two or three years having convinced me of the importance of increasing our knowledge of the highest development to which the sporophyte attains in the Muscineae; this is undoubtedly reached by *S. luteum* and its immediate allies.

In the sporophyte of *Splachnum luteum* we have a structure with a remarkable similarity to an umbrella, the handle end of which is inserted in the tissues of the oophyte and is known as the foot. The seta is much elongated, bearing the umbrella-like expansion, the apophysis, at the top just below the sporangium. It is the structure of the apophysis with which we are chiefly concerned.

A transverse section through the vaginula, including the foot of the sporophyte, shews the tissues of the oophyte in this part to contain a considerable quantity of organic substance, and this

[1] Beit. zur Anatomie und Physiologie der Laubmoose: *Jahrb. für wissen. Bot.* Bd. XVII.

[2] Sur les Homologies des Mousses. *Bulletin de la Société des Sciences de Nancy*, Fasc. XIX. 1886.

[3] On the Anatomy and Development of the Sporogonium of the Mosses. *Journ. Linn. Soc. Bot.* vol. XXIV.

is seen to be more particularly the case in the layers of cells next the foot. The foot itself is seen to consist of a cylindrical mass of parenchyma with an external layer of epidermal cells of a somewhat columnar form, which contain a considerable quantity of protoplasm and contain large distinct nuclei. The protoplasm of these cells is found to be aggregated towards the external surface, the nucleus being usually found in the peripheral mass of protoplasm; a thin layer of protoplasm is formed all round the other walls of these cells and large vacuoles are present which are traversed by fine protoplasmic filaments. There are in these cells a great number of small round protoplasmic bodies, especially aggregated round the nucleus and towards the peripheral side of the epidermal cells, which appear to be a kind of plastid. As plastids are usually employed in some absorptive and assimilative process I would suggest that in this case the plastids are engaged in the process of absorbing nutriment for the sporophyte from the tissues of the oophyte.

In the centre of the foot there is a definite central strand consisting of two kinds of tissue, an outer layer of cells containing protoplasm, the leptophloëm[1] surrounding an inner strand of more thin-walled cells containing no protoplasm, the leptoxylem. The leptophloëm cells have their protoplasm aggregated towards the periphery like the epidermal cells, but differ from them in that they contain no plastids. The leptoxylem I have proved[2] in other species of *Splachnum* to be the tissue by means of which water is conveyed up the seta to the apophysis.

The seta has a distinct epidermis, beneath which there is a layer of sclerotic supporting tissue and parenchyma; this parenchyma together with the sclerotic tissue forms the cortex. In the centre is the central strand which in the lower part has almost the same structure as that above described in the foot, except that it is larger and is less distinctly delimited from the surrounding tissue. Higher up in the seta the centre of the leptoxylem is occupied by a large intercellular space which forms an intercellular passage for nearly the whole length of the seta. This intercellular space is lysigenous in origin. A similar passage or canal is found in several other species.

A longitudinal median section through the umbrella-shaped apophysis shews that the "central strand" here swells out into a large pear-shaped mass of cells which in the mature sporophyte contain no protoplasm, and in the younger states only a very small quantity with small inconspicuous nuclei. Chlorophyll bodies are absent except in the two outermost layers of cells, even in the youngest specimens observed, and here there are only a very few; the cells are all thin walled and cubical in shape, without any

[1] Cf. Vaizey, *loc. cit.* for definition of the terms leptophloëm and leptoxylem.
[2] Vaizey, Note on the Transpiration of the Sporophore of the Musci. *Annals of Botany*, vol. I.

intercellular spaces between them. In this tissue, which may be regarded as an aqueous tissue, large masses of crystals of calcium oxalate were frequently found.

Outside the aqueous tissue there is a quantity of parenchymatous tissue with numbers of schizogenous intercellular spaces; the cells all contain large numbers of chlorophyll bodies, and this tissue extends into the umbrella-shaped organ. On the upper surface the cells are arranged close to one another, and shew a distinct tendency to an elongation of their axes in a direction vertical to the upper surface, thus forming a tissue with a striking likeness to the pallisade tissue of the leaves of the higher plants[1]; this is rendered more striking by a comparison with the development of the parenchyma of the lower surface, where the cells are very much elongated in a direction parallel to the surface with very much larger intercellular spaces. Stomata are found largely on the upper surface but none occur on the lower. A large quantity of starch is formed in the apophysis while it is quite young and immediately after it first becomes umbrella-shaped and before the spores ripen, the apophysis at this time being green; at a later stage the starch disappears and the starch-forming plastids which before were large and well formed degenerate into small and comparatively inconspicuous bodies, the starch apparently being used up in the formation of spores. The apophysis then becomes the characteristic yellow colour of the species.

The apophysis seems to be therefore comparable in many ways to a leaf of the Vascular Plants, and possibly homologous with true leaves. The discussion of this point must however be postponed for the present.

The mode of life of the sporophyte may be summarised thus:—

The *foot* absorbs all the substances requisite for the nourishment of the young embryo until at any rate the apophysis is developed and throughout the life of the sporophyte water, and in all probability some inorganic substances.

The *apophysis* as soon as it is developed forms large quantities of carbohydrate, thus rendering it unnecessary for carbohydrate substances to be absorbed from the oophyte, which would hardly be able to supply sufficient for the development of so large a structure as the sporophyte. So by this means the oophyte is preserved from destruction by the sporophyte, so that the sporophyte is prevented from cutting off its own means of obtaining water.

The central strand of the *seta* is the channel by means of which the water absorbed by the foot is conveyed to the apophysis.

[1] Haberlandt, *loc. cit.*, also makes a comparison between the chlorophyll containing tissue of the sporophyte of the mosses and the pallisade tissue of leaves of vascular plants, but none of the cases he investigated are as striking as that of *Splachnum luteum*.

(5) *Note on the Germination of the Seeds in the genus* Iris. By M. C. POTTER, M.A., Peterhouse.

IT is a well-known fact that some seeds germinate very soon after being planted, while others take some considerable time. This is very well marked in the seeds of the various species of Iris, the seeds of some species taking only a few weeks, those of other species requiring twelve or even more months to germinate. In order to try to discover the reason of this, seeds of various species were planted in ordinary flower-pots, and exposed to the same conditions of temperature and moisture, and the embryos examined from time to time. The embryos were taken from the seeds, the protoplasm fixed with picric, chromic and osmic acids, then cut and mounted in the usual way.

When the seed is ready to be detached from the parent plant we find the embryo to be fully developed as regards its size and morphological differentiation, but unfit to germinate until important changes have taken place in the nucleus and protoplasm of the cells. In this stage the cells of the embryo are densely filled with nucleus and protoplasm, but contain no vacuoles; but gradually, as the seed is getting ready to germinate, small bodies are formed in the protoplasm which increase in size, are numerous in each cell, and are found to consist of proteid matter. Each of these bodies is included in a vacuole. Each cell therefore now has its nucleus and protoplasm containing numerous vacuoles with their proteid bodies. These bodies disappear on germination, and hence, since different seeds take different lengths of time to effect these changes, they must necessarily germinate at different intervals of time. The species of Iris which germinate quickly perform these changes in a short space of time, while the long germinating ones take a considerable time.

The Iris seeds are endospermous, with the embryo entirely enclosed in the endosperm, but the cells which immediately cover the radicle are few and form a kind of cap, which must be removed before germination. When germination commences the cotyledon elongates and pushes the radicle and plumule outside the seed to a greater or less distance, according to the nature of the soil, so that the most favourable place can be found; part of the cotyledon remains in the seed in order to transfer the contents of the endosperm to the young plant. The diameter of the hole in the endosperm through which the radicle is pushed always remains small in size, so that as the parts of the cotyledon on each side of it grow, a constriction is here formed in the cotyledon.

In seeds which have the cells covering the radicle removed we find often the cotyledon much elongated, and, if the seed lies exposed or nearly so, becoming spirally twisted, so that the radicle may even be directed upwards and the free end of the cotyledon does not remain in close contact with the endosperm; hence the

young plants may die, or are weakly and feeble. It would thus appear that the use of the constriction above mentioned is to keep the cotyledon closely pressed to the endosperm so that the latter may be transferred to the young plant;—for the excessive elongation above mentioned would ensure this. The small diameter of the hole acts therefore as a resistance and directs the growth of the young plant. Seeds buried in the soil do not shew this markedly because the soil naturally offers some resistance to the cotyledon. A similar constriction is found in the germination of the Date-Palm.

(6) *On the protection afforded by the stipules to the buds of* Betula Nana. By M. C. POTTER, M.A., Peterhouse.

November 26, 1888.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following communications were made:

(1) *On Solution and Crystallization*, Part II. By Professor LIVEING.

Abstract.

IN this paper the author gives a physical explanation of the causes of the molecular arrangements which give rise, as was shewn in the former paper on this subject, to the external forms and cleavages of crystals. No peculiar force is required to produce these arrangements of the molecules provided they attract each other. The supposition made is that while the parts of the molecules are in constant motion, the excursions of the parts of each molecule from its centre of mass are limited in the solid state. The extent of these excursions will in general be different in different directions, so that the molecule will on the average be comprised within an ellipsoid. For the same kind of matter it is supposed that these ellipsoids are all similar and equal. If the molecules attract one another, the problem of their arrangement in stable equilibrium becomes the problem of packing the greatest number of similar and equal ellipsoids in a given volume. The solution of this problem is given, and it is shewn that the ellipsoids will be all similarly situated, though the orientation of their axes is a matter of indifference. The matter of each molecule will not in general be uniformly distributed within the molecular volume, and if in any case it be chiefly massed about a particular plane or a particular line, this massing

will in general determine the orientation of the axes of the ellipsoids. It is then shewn how the arrangements of the ellipsoids are related to the crystalline forms. If the three axes of the ellipsoids be all unequal the crystal will be anorthic, oblique, or right prismatic, according to the orientation of the axes. If two of the axes be equal so that the ellipsoids become spheroids, the crystal will be rhombohedral or pyramidal, according to the orientation of the axis of revolution of the spheroid, or in the case of special relations between the axes, cubic with cubic cleavage, or cubic with dodecahedral cleavage. If the three axes be all equal the crystal will be always cubic, but with a predominant octahedral cleavage.

(2) *On the metameric transformation of Ammonium Cyanate.* By H. J. H. FENTON, M.A., Christ's College.

WITH a view of investigating the class of changes known as isomeric and metameric, the author has prepared ammonium cyanate in a pure state, and has studied the conditions of its transformation into urea. The results indicate that the change proceeds rapidly at first, then becomes slower, and ultimately reaches a limit which is a function of the temperature. In no case examined was the transformation completed: even at 100° C. for 20 hours only 88 per cent. was transformed.

Conversely, experiments render it highly probable that urea in solution is, in part, re-transformed into ammonium cyanate, so that the phenomena may to some extent be compared with the allotropic changes of phosphorus, &c.

(3) *On the Iodides of Copper.* By D. J. CARNEGIE, B.A., Gonville and Caius College.

SOLUTIONS of CuI_2 containing about ·3 gram in 100 c.c. have been obtained. The author's experiments render it very probable that when potassium iodide is added to an aqueous solution of copper sulphate, in molecular proportions, cupric iodide is produced but the whole of the potassium iodide is not decomposed, and that the remaining KI reacts with the CuI_2 produced to form CuI and $KI . xI$.

(4) *On a Compound of Boron Oxide with Sulphuric Anhydride.* By R. F. D'ARCY, B.A., Gonville and Caius College.

BY the reaction of boric acid with sulphuric anhydride, or with concentrated sulphuric acid containing a large quantity of sulphuric anhydride, a definite compound having the composition $BH_3(SO_4)_3$ has been obtained.

(5) *Experiments on Colour-Perception; and on a Photo-voltaic theory of Vision.* By CHARLES V. BURTON (B.Sc. London).

I. *Experiments on Colour-Perception.*

IT appears as yet to be an undecided question, whether or not the violet of the spectrum is physiologically redder than the blue.

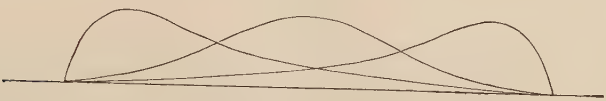
The present paper contains an account of some subjective experiments on colour-perception which were made at the Cavendish Laboratory, chiefly with the object of putting this question to the test. A spectroscope with two prisms was arranged so as to give a bright solar spectrum. On looking with one eye at a given part of the spectrum for about two minutes, the eye became fatigued, so that its sensitiveness to certain rays was diminished, and on turning to another part of the spectrum, a greater or smaller change of intensity and tint was perceptible, depending on the physiological relation of the colour now observed to that which had produced retinal fatigue. The observation was assisted by using the untired eye as a standard for comparison. In place of looking at a selected portion of the solar spectrum, coloured glasses were usually employed to sift the light of the sun, the colours transmitted by them being of sufficient intensity to affect the eye very powerfully. The fatiguing was continued for about two minutes at a time, and was constantly repeated during the observations, which were now made with a 4-prism spectroscope, so that only a limited range of colour was included in the field of view. A preliminary test having shown that the two eyes were very nearly alike in their perception of colours, and in their change of sensitiveness when fatigued for red, the fatigue was always subsequently produced in the right eye, and the left used for purposes of comparison.

After looking at the sun through "red" glass (which transmits red, and orange, and a very little green), there was an apparent change of tint in the solar spectrum extending as far as the greenish-blue, the blue not being appreciably affected; while on the other hand the violet looked very much bluer to the right eye and by contrast redder to the left. When the right eye was tired for green light, the yellow and yellowish-green appeared more orange, the green paler, and the greenish-blue bluer, while deep blue and violet were hardly affected. On tiring the right eye for blue and violet no change of appearance was observed in red, orange or yellow. Grass-green looked slightly paler and yellowish, bluish-green and greenish-blue looked greener and paler, the blue duller and paler, and the violet distinctly redder. It was often found that the impressions of the left eye were

changed, as well as those of the right, though usually to a smaller extent.

A diagram of colour sensations is usually given as in fig. 1; each sensation being affected by the whole range of the visible

Fig. 1.



spectrum. So far as these observations go, however, they seem to indicate that, with the exception of violet, rays of higher refrangibility than the greenish-blue do not affect the "red" colour-sensation. The "green" sensation seems to be affected through a range extending from orange to blue, the lower limit for the "violet" sensation being somewhere in the green or yellowish-green.

It also seems that the "green" sensation participates only to a very small extent in the perception of blue; this colour being distinguished from violet by not exciting the "red" colour sensation.

It may here be noticed that a pure and uniformly illuminated diffraction spectrum would appear throughout of equal brightness, and its tint would appear to change at a uniform rate if the colour-sensations followed the law indicated in fig. 2; the "green" sensation, however, being twice as strongly affected by white light as either of the others. If the straight lines in fig. 2 be replaced by more continuous curves as in fig. 3, there will be a variation of apparent intensity in different parts of the spectrum, and a still greater variation in rate-of-change-of-tint, which now becomes zero at the middle and ends of the spectrum, and attains a maximum at two intermediate points.

It will be clear that the arrangement indicated in fig. (2) or in fig. (3), where each simple colour affects only two sensations,

Fig. 2.

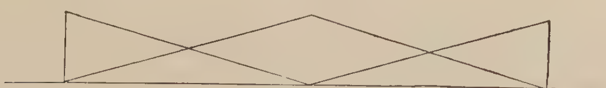


Fig. 3.

will give greater purity of tint than would be possible with the arrangement of fig. (1), where all three colour-sensations are always affected; for in the latter case, a certain portion of the

resulting impression may be analytically assigned to white light. The present observations seem to indicate that the curves for the two extreme sensations overlap in the middle of the spectrum, so that all three sensations participate in the perception of colours from the yellowish-green to the greenish-blue, whilst not one of the three sensations is sensitive throughout the whole range of the visible spectrum.

One or two matters of detail should here be mentioned. It was always found that for a considerable time after the right eye had been fatigued for a given colour, there was a marked difference in the appearance of surrounding objects as seen by the two eyes, whilst on closing both eyes, no colours were seen, showing that the change effected by fatigue was merely one of sensitiveness, and did not involve any persistence of active disturbance. Before tiring the eye with another colour, an interval of some hours was always allowed to elapse, so that vision had again become normal.

Whilst looking at the unclouded sun through red glass there was a gradual alteration in the appearance of the sky, which seemed to change from a deep brick-red to a dull foggy-yellow. Similar but less marked effects were obtained while looking through green or blue glass. No doubt these changes of tint are accounted for by the fact that the colour-sensation chiefly involved becomes much more completely and rapidly fatigued than that which is only moderately excited.

On looking for some time at the violet end of the spectrum, it appears redder to the tired eye. Again spectrum-violet diluted with white can be exactly matched by a mixture of blue and extreme red. Two possible explanations present themselves.

(1) That the apparatus which is chiefly sensitive to red impressions attains a secondary maximum of excitability in the neighbourhood of the violet.

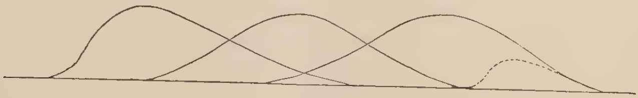
(2) That violet light is partly changed to red by fluorescence of the retina, and so affects the red colour-sensation.

The latter view was experimentally tested in the following way. A beam of sunlight passed through a hole in a shutter which was covered with a double thickness of cobalt glass. This beam was received on the concave mirror of an ophthalmoscope, and being concentrated on the eye of the subject, the orange-red retina could be distinctly seen, as in the ordinary use of the instrument with white light. The appearance was so bright and the hue so orange a red as to leave no doubt that the effect was mainly due to fluorescence. The double thickness of cobalt glass transmits none of the less refrangible rays, except a dim and narrow band in the extreme red.

Fig. (4) roughly indicates the general nature of the results obtained; the dotted line corresponding to the excitement of the red sensation by fluorescence from violet light.

My best thanks are due to Prof. J. J. Thomson and Mr Glazebrook for much kind assistance and advice, and also to my friends Mr Jones and Mr Masom, who were good enough to place their eyes at my disposal for the experiment on fluorescence.

Fig. 4.



II. *On a Photo-Voltaic Theory of Vision.*

Prof. Dewar and Dr M^cKendrick have shown[1] that there is a considerable E. M. F. between the anterior portions of the eye and a transverse section of the optic nerve. Making contact by means of unpolarisable electrodes, a current was led round the coils of a galvanometer. It was found that this current was altered in amount when light fell on the retina, but not when light fell on other parts of the eye, such as the optic nerve.

It has further been shown by numerous experimenters, including Ritter, Helmholtz and Du Bois-Reymond, that a current in the optic nerve produces the sensation of light. The question then is, by what means does light falling on the retina produce or modify the current in the optic nerve? On comparing the laws of perception of light with the photo-voltaic properties of selenium as determined by Prof. W. G. Adams and Mr Day[2], there are found to be some striking resemblances, though of course selenium can hardly be the substance to which the retina owes its sensitiveness. It may be as well here to enumerate some of the properties of selenium, and then to point out their analogy to the laws of light-perception.

(1) When a current flows in selenium it produces a twofold effect; (*a*) an increase of resistance in the direction of the current and a decrease in the opposite direction, (*b*) an opposing E. M. F. due to some kind of polarization. Both these effects persist for an appreciable time after the current has ceased.

(2) When light falls on selenium it causes (*a*) a decrease of resistance which persists for a short time after the light is shut off; (*b*) an E. M. F. which shows neither lag nor persistence in any appreciable degree; (*c*) fatigue, which causes this E. M. F. to fall off rapidly as the action of light is continued, and which is in some measure *selective:* the selenium showing most fatigue for the colour to which it has been exposed.

(3) The rays which affect the electrical properties of selenium are precisely the visible rays; the ultra-red and ultra-

[1] *Edin. Phil. Trans.* XXVII. 141.
[2] *Proc. R. S.* XXIII. 535; XXIV. 163; XXV. 113.

violet rays producing little if any effect. Dewar and M[c]Kendrick also find that the visible rays are those which affect the current obtained from the eye; and in each case it is the yellow and green rays which are the most effective.

They appear to ascribe the variations of current which they obtained to E. M. F.'s arising from a chemical action which is produced in the retina by the impact of light. They further suppose that when the eye is in darkness there is a certain current in the optic nerve, which changes in strength when light falling on the retina produces an additional E. M. F. If, however, as would appear from the experiments of Helmholtz and others, the currents in the optic nerve really determine the sensations of vision, it seems probable that the current corresponding to any position on the retina travels to and from the brain by nearly coincident paths; (for if the currents from different parts of the retina followed a common path for any part of their course they would mutually interfere with one another). In that case, the E. M. F. along the optic nerve will not produce any current in the circuit, and the currents producing visual impressions will be entirely due to E. M. F.'s set up by the action of light on the retina.

Now we have seen (2, *b*) that in the case of selenium the photo-voltaic E. M. F. follows changes of illumination almost instantaneously, just as visual impressions do; and again light falling on selenium produces fatigue, just as it does on the retina. In both cases too the fatigue is selective, though in the eye this is accounted for by the independence of the colour-sensations.

As regards persistence of impressions, we know that if light falling on selenium produces a current of sufficient strength, this current will set up a polarizing E. M. F., so that when the light is removed there is a reversed current. From the memoir of Adams and Day, it does not appear that they observed any further persistent E. M. F. due to the action of light. If we suppose the effect of light on the retina to be of a photo-chemical nature, it is difficult to see how sensitiveness is maintained. In order always to keep fresh liquid in contact with the retina, there would have to be enormously rapid diffusion; and then too (as appears from the laws of colour-perception) there must be three sensitive compounds in the eye, each affected by a different range of colour, and each when acted on by light affecting a different portion of the retina. If the sensitive compounds are supposed to be contained in the retina itself, it seems impossible that they should be renewed with sufficient rapidity to maintain sensitiveness.

Following the varying effects which Dewar and M[c]Kendrick obtained by the action of light on the eyes of different animals, I find they can all be explained on the supposition that the

action is photo-voltaic; both a transitory E. M. F. *opposed* to the "normal" current, and a persistent diminution of resistance being produced. The physiological conditions however were necessarily complex, so these explanations are not given in detail, as it would perhaps be explaining too much.

On the whole the most striking analogy between vision and photo-voltaic action is the range of colour which produces the action in each case. Supposing vision to be due to photo-chemical action, it has always been a paradox to see why the less refrangible end of the spectrum should be visible as far as the red, while the powerfully actinic rays beyond the violet are almost entirely non-luminous.

(6) *On the Geometrical Interpretation of the Singular Points of an Equipotential System of Curves.* By J. BRILL, M.A., St John's College.

1. In a paper presented to the Society during the Lent Term I called attention to a paper published by Siebeck in the fifty-fifth volume of *Crelle.* In that paper he enunciates a theorem which is equivalent to the statement that all equipotential systems of curves are confocal, the branch points of the system being the common foci of the constituent curves of the system. This statement, as I pointed out, is contradicted by the well-known case of a system of coaxal circles. By studying this case I have been enabled to find out where Siebeck went wrong*, and also to obtain a geometrical interpretation of the branch points of an equipotential system of algebraical curves. And, in what follows, I have also shewn that this interpretation gives promise of capability of development into a method which would enable us to discover all the possible equipotential families consisting of algebraical curves of a given order.

2. In seeking the intersections of real algebraical curves imaginary values turn up in pairs. Thus if

$$x = a + ib, \quad y = c + id$$

be the coordinates of one of the points of intersection, there will be a corresponding point of intersection whose coordinates are

$$x = a - ib, \quad y = c - id.$$

We shall call these points respectively A and B. We shall also denote by the symbol I that circular point at infinity that lies on the line $x + iy = 0$, and by the symbol J that one that lies on the line $x - iy = 0$. Then the equations of the lines

* Kummer appears to have fallen into an error similar to that of Siebeck: see a paper in the thirty-fifth volume of *Crelle*, entitled "Ueber Systeme von Curven welche einander überall rechtwinklig durchschneiden."

$$AI, . AJ, \quad BI, \quad BJ$$

are respectively

$$x+iy=a-d+i(b+c),$$
$$x-iy=a+d+i(b-c),$$
$$x+iy=a+d-i(b-c),$$
$$x-iy=a-d-i(b+c).$$

The coordinates of the point of intersection of AI and BJ are

$$x=a-d, \quad y=b+c,$$

and those of the point of intersection of AJ and BI are

$$x=a+d, \quad y=-(b-c).$$

The remaining points of intersection of the lines AI, AJ, BI, BJ are in general imaginary, as also are the intersections of these lines with those obtained by joining a second similar pair with the circular points at infinity.

Call the two real points of intersection P and Q; and denote the coordinates of the imaginary points A and B by the symbols (α_1, β_1) and (α_2, β_2). Let z_1 and z_2 be the complex quantities that correspond to the points P and Q according to Gauss's method of representing complex quantities by points. Then we have

$$z_1=a-d+i(b+c)=a+ib+i(c+id)=\alpha_1+i\beta_1,$$

and similarly

$$z_2=a+d-i(b-c)=a-ib+i(c-id)=\alpha_2+i\beta_2.$$

3. Suppose that we have an equipotential system of curves given by the equation $z=f(w)=f(\xi+i\eta)$. Also let

$$f(\xi+i\eta)=\phi(\xi,\eta)+i\psi(\xi,\eta),$$

so that

$$x=\phi(\xi,\eta) \quad \text{and} \quad y=\psi(\xi,\eta).$$

To find the points in which the curve $\eta=\alpha$ is cut by the consecutive curve $\eta=\alpha+d\alpha$, we have the equation

$$\begin{aligned} f(\xi+i\alpha) &= f\{\xi+i(\alpha+d\alpha)\} \\ &= f(\xi+i\alpha)+id\alpha f'(\xi+i\alpha). \end{aligned}$$

Thus the requisite values of ξ are given by the equation

$$f'(\xi+i\alpha)=0.$$

Now let $\lambda+i\mu$ be a root of the equation $f'(w)=0$. Then we have

$$\xi+i\alpha=\lambda+i\mu=\lambda+i(\mu-\alpha)+i\alpha.$$

Thus one of the points of intersection will be the point whose coordinates are

$$\xi = \lambda + i(\mu - \alpha), \quad \eta = \alpha,$$

and this point will be in general imaginary.

Suppose that one of the families of the system consists of algebraical curves having no real points of intersection. The number of points of intersection will be even, and it will be possible to arrange the points in pairs similar to that discussed in the preceding article. Further we see by that article that if we join the points of intersection with the circular points at infinity, the number of real intersections of the lines so formed will be equal to the number of intersections of the curves. And, if (ξ_1, η_1), (ξ_2, η_2), &c., be the coordinates of the points of intersection of the curves, then the z's of the said real points will be

$$\phi(\xi_1, \eta_1) + i\psi(\xi_1, \eta_1) = f(\xi_1 + i\eta_1),$$
$$\phi(\xi_2, \eta_2) + i\psi(\xi_2, \eta_2) = f(\xi_2 + i\eta_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus the z's of the above-mentioned points are obtained by substituting the roots of the equation $f'(w) = 0$ in the expression $f(w)$. In other words they are the branch points of the system.

4. In the preceding article we obtained a geometrical interpretation of the branch points of an equipotential family of algebraical curves having no real points of intersection, and we now proceed to the discussion of the different cases that may arise. We have proved that the locus of ultimate intersections of the family of curves consists of the collection of straight lines joining the branch points of the system to the circular points at infinity. Should this collection of straight lines form a proper envelope of the family, then the curves are confocal, the branch points being the common foci of the members of the family. There are a large number of known instances of confocal systems of equipotential curves.

A second possible case is when all the curves of the family pass through a fixed set of imaginary points. An instance of this is the above-mentioned case of a set of coaxal circles. It will be found in this case that the real intersections of the lines joining the imaginary points of intersection to the circular points at infinity are the limiting points of the system. This may be easily proved directly, or it may be deduced from the harmonic properties of the complete quadrilateral obtained by joining the four points of intersection of two conics.

In other cases the straight lines joining the branch points to the circular points at infinity may constitute a node locus or a cusp locus, or a locus of singularities of a higher order. In special cases these lines may coincide with the tangents at the nodes or cusps and we again obtain foci.

In some cases a portion of the system of lines we are considering may constitute a proper envelope, while the remaining portion comes under one or more of the other cases we have mentioned. In this case the curves of the system will have only a certain number of their foci coincident.

5. In Article 3, we supposed that the successive curves of the family in question did not intersect. If however the curve $\eta = \alpha$ cut the curve $\eta = \alpha + d\alpha$, then the equation $f'(\xi + i\alpha) = 0$ will be satisfied by real values of ξ, and it is evident that the two curves will intersect in branch points. Thus if each curve of the family cut the consecutive one, all the curves of the family will pass through a set of fixed points, these points being branch points of the system to which the family belongs. The set of fixed points through which the curves of the family pass need not however include all the branch points of the system.

It is easy to shew that an equipotential family of curves cannot have a real envelope which does not consist of a system of discrete points. We have shewn that if two consecutive curves of the family intersect, they intersect in branch points; and it remains to be proved that these branch points are discrete, and do not form a locus. Let

$$f'(\xi + i\eta) = f_1(\xi, \eta) + if_2(\xi, \eta).$$

Then in order that $f'(w) = 0$, we must have

$$f_1(\xi, \eta) = 0, \text{ and } f_2(\xi, \eta) = 0.$$

These equations represent two curves in the w plane which are subject to the condition that they should intersect orthogonally. Hence their points of intersection must be a system of discrete points; as these points could only form a locus in case the two curves should coincide, which is impossible. Thus the branch points of the family, which are the corresponding points in the z plane, also form a discrete set.

There is however one special case in which the members of an equipotential family of curves may have a real envelope which is not a point. This occurs when all the members of the family touch the line at infinity. It is also conceivable that the members of an equipotential family may have the line at infinity for a node locus, a cusp locus, or a locus of singularities of a higher order.

Other special peculiarities may arise through infinity turning up as a branch point. An instance of this will be given further on.

6. We now proceed to illustrate the above theory by the discussion of some particular cases, taking first that of a family of straight lines. A straight line has no singular points, and two real straight lines always meet in a real point. Thus it is

evident that the only possible case is that of a family of straight lines passing through a fixed point. The case of a family of parallel straight lines is of course included in this.

7. Passing on to curves of the second order we have the well-known case of a system containing two families of confocal conics, the one family consisting of ellipses and the other of hyperbolas. Under this are included two particular cases. In the first of these one of the foci passes off to infinity and we have two families of confocal coaxal parabolas. In the second the foci coincide and we have a family of concentric circles with the orthogonal family of straight lines.

Since proper curves of the second order have no singular points, it is evident that in order to arrive at a complete enumeration of the equipotential families of the second order, if we leave parabolas out of account, we have only two more cases to examine. Firstly, the case of a family of conics having one common focus and passing through two fixed points, and secondly the case of a family of conics passing through four fixed points. It is understood that the points of intersection are not necessarily real. There is no known family belonging to the first of these cases, but there are several families belonging to the second.

In the first place we have the system consisting of two families of coaxal circles. The circles belonging to one family intersect in two real points, viz. the limiting points, and in two imaginary points, viz. the circular points at infinity. The circles belonging to the other family intersect in four imaginary points, viz. the circular points at infinity and two imaginary points lying on the radical axis. A special case of this is the system consisting of two orthogonal families of circles, all the circles of each family touching one another at the origin.

Secondly, we have the system consisting of two orthogonal families of rectangular hyperbolas. In this case all the curves of each family have double contact at infinity. There is, however, a peculiarity about this case. The points in which the members of one of the families intersect are not coincident with those in which the members of the other family intersect. This peculiarity arises from the fact that we have not certain discrete points at infinity for singular points, but infinity itself turns up as a singular point.

Thirdly, we have the family of rectangular hyperbolas that consists of the orthogonal trajectories of a family of confocal Cassinians. These hyperbolas intersect in two real points, viz. the common foci of the Cassinians, and in two imaginary points lying on the line which bisects at right angles the line joining the said foci.

There is one more known family of equipotential curves of the second order. It consists of a series of parabolas having a

common focus and passing through a fixed point. And, since it is possible to draw two parabolas having a given focus and passing through two fixed points, it is evident that we shall obtain two sets of parabolas satisfying the given conditions. These two sets must however be considered as two distinct families of equipotential curves*.

8. We may remark that it is theoretically possible to discover all the families of equipotential curves that belong to the cases enumerated in the preceding article. For example, take the case of a family of conics passing through four fixed points. The most general form of the equation representing such a family is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda (Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C) = 0,$$

where λ is the parameter of the family, and the other coefficients are constants. The condition that this may represent an equipotential family of curves is that the expression

$$\left\{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right\} \Big/ \left\{\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right\}$$

may be a function of λ. If the value of this expression be calculated, it will be found that the numerator and denominator are of the sixth degree in x and y, and consequently the said condition will be of the form

$$\frac{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2}{\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}} = \frac{p\lambda^3 + q\lambda^2 + r\lambda + s}{P\lambda^3 + Q\lambda^2 + R\lambda + S}.$$

Substituting for λ, reducing, equating coefficients, and eliminating p, q, r, s, P, Q, R, S from the resulting equations, we should at length obtain the necessary relations that must exist among the constants $a, b, c, f, g, h, A, B, C, F, G, H$ in order that the family of conics may be equipotential.

Similarly it would be theoretically possible to discover if there were any equipotential families consisting of conics having a common focus and passing through two fixed points. We could also test the remaining case of parabolas, viz. a family of parabolas passing through three fixed points.

It is to be observed that the work for discovering the equipotential systems among the higher plane curves is also theoreti-

* This case may be derived from that of a family of straight lines passing through a point by means of the transformation $w^2 = az$, the straight lines lying in the w plane and the parabolas in the z plane. If we look at the case from Riemann's point of view, we must consider the two sets of parabolas as traced upon different sheets of the Riemann's surface belonging to the given transformation.

cally possible. Node loci, cusp loci, &c., would however now become admissible, and the number and complication of the cases to be discussed would rapidly increase with the order of the curves. It is to be hoped that further research will bring to light more geometrical considerations which will enable us successfully to track down the various equipotential systems of algebraical curves. One thing to be greatly desired is a purely geometrical statement of the criterion to be applied to a family of curves in order to discover whether they form an equipotential family.

We may remark that, in the enumeration of cases to be discussed, some of the more special forms of the curves of a given order would give the most trouble.

In a paper in the seventy-seventh volume of *Crelle* entitled "Ueber ebene algebraische Isothermen," Schwarz has proved that all equipotential systems consisting of algebraical curves may be derived, by means of a transformation deduced from an algebraical function, from one of three cases. These cases are: (1) the system consisting of two families of parallel straight lines, (2) the system consisting of a family of straight lines passing through a point and the orthogonal family of concentric circles, and (3) a system consisting of two orthogonal families of curves known as Siebeck's curves, which may be derived from the system consisting of two orthogonal families of parallel straight lines by means of the transformation $z = c \operatorname{sn} w$.

In the bibliography at the end of the twelfth chapter of his "Isogonale Verwandtschaften," Holzmüller gives the following reference.

Hans Meyer—Ueber die von geraden Linien und von Kegelschnitten gebildeten Schaaren von Isothermen, sowie über einige von speciellen Curven dritter Ordnung gebildeten Schaaren von Isothermen. Inauguraldissertation, Göttingen 1879.

Holzmüller also remarks that this memoir solves the problem of determining a family of equipotential curves from two given consecutive curves of the family, deals with certain special cases, and determines all rational functions by means of which a family of parallel straight lines can be transformed into a family of cubics. Judging from the title of this memoir and from Holzmüller's remarks, it would seem that Meyer's work is closely allied to the subject in hand. I have not, however, as yet been able to see a copy of this paper.

In conclusion I may state that I have been unable to discover any cases involving node loci or cusp loci, but I do not see any *a priori* reason why they should not arise. Several more cases might have been given of equipotential families of algebraical curves passing through a system of fixed imaginary points.

[Since the paper was read I have seen Meyer's dissertation.

He gives the same enumeration of equipotential families consisting of curves of the second order as I have done in Art. 7, and proves that it is complete. All the families of equipotential cubics discussed by him consist of cubics passing through nine fixed points.

I find that the enumeration of the equipotential families of conics had been given by Vonder Mühll in the sixty-ninth volume of *Crelle;* but his work is extremely complicated.—*Dec.* 18, 1888.]

COUNCIL FOR 1888—9.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

January 28, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

E. H. Douty, M.A., King's College,
H. W. Richmond, M.A., King's College,
G. F. C. Searle, B.A., Peterhouse.

The following Communications were made:

(1) *The application of the theory of the Transmission of Alternating Currents along a wire to the Telephone.* By J. J. THOMSON, M.A., F.R.S., Cavendish Professor of Experimental Physics, Cambridge.

IN a paper on "Electrical Oscillations on Cylindrical Conductors," *Proc. Lond. Math. Soc.*, vol. 17, p. 320, I have shown that if we have an alternating current travelling longitudinally along a thin wire and varying as $\epsilon^{i(mz+pt)}$ where the axis of the wire is taken as the axis of z, and if

$$k^2 = m^2 - \frac{p^2}{V^2},$$

$$n^2 = m^2 + \frac{4\pi\mu i p}{\sigma},$$

where σ is the specific resistance of the wire, μ its magnetic permeability, then

$$k^2 = \frac{cp^2}{V^2} \frac{\mu n}{m^2 - n^2} \frac{1}{a} \cdot \frac{J_0(ina)^*}{J_0'(ina)} \quad \text{..................(1)},$$

where c is the electrostatic measure of the capacity of unit length of the wire, V the velocity of propagation of electro dynamic action, and $J_0(x)$ denotes the Bessel's function of zero order which is not infinite when $x = 0$.

In deducing the above equation Maxwell's theory of electric action has been assumed. We must consider different cases of equation (1) corresponding to different values of na and c.

Case I. na exceedingly small. Here

$$J_0(ina) = 1, \quad J_0'(ina) = -\tfrac{1}{2} ina,$$

so that equation (1) becomes

$$k^2 = -\frac{ip^2 c\sigma}{2\pi a^2 p v^2},$$

or

$$m^2 = \frac{p^2}{V^2}\left\{1 - \frac{ic\sigma}{2\pi a^2 p}\right\} \quad \text{......................(2)}.$$

From the following table we see that this formula will be approximately true whenever $4\pi\mu p a^2/\sigma$ is less than 5, as it only involves the assumption that $inaJ_0(ina)/J_0'(ina)$ is approximately equal to 2. Unless p is of the order 10^{10} we may put

$$n^2 = \frac{4\pi\mu i p}{\sigma};$$

Value of $4\pi p a^2/\sigma$	Value of $inaJ_0(ina)/J_0'(ina)$
1	$2\left(1 - \frac{i}{8}\right)$,
2	$2\left(1 - \frac{i}{4}\right)$,
3	$2\left(\frac{6}{7} - \frac{3}{7}i\right)$,
4	$2\left(\frac{3}{4} - \frac{5}{8}i\right)$.

* The factor μ is omitted in the paper quoted.

In the case we are considering we shall assume that $4\pi pa^2/\sigma$ is so small that $c/4\pi pa^2/\sigma$ is large. We must remark that c cannot be made to fall below a not very small fraction, for that even when there is no conductor at less than an infinite distance from the wire, the wire will behave (see *Proc. Lond. Math. Soc.* p. 315) as if

$$c = \frac{1}{2} \frac{1}{\log p\sigma\gamma^2/\pi V^2},$$

where $\log \gamma = \cdot 577 - \log 2$.

Since in this case the second term in the bracket on the left-hand side of equation (2) is much larger than the first, we have

$$m^2 = -\frac{ip^2}{V^2} \frac{c\sigma}{2\pi a^2 p},$$

$$m = \frac{1}{V} \left\{\frac{c\sigma p}{2\pi a^2}\right\}^{\frac{1}{2}} \left\{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right\},$$

this represents a disturbance, propagated with the velocity

$$V \left\{\frac{4\pi a^2 \delta}{c\sigma p}\right\}^{\frac{1}{2}},$$

and fading away to $1/e$ of its original value after traversing a distance

$$V \left\{\frac{4\pi a^2}{c\sigma p}\right\}^{\frac{1}{2}}.$$

Thus in this case both the velocity of propagation of disturbances and the rate at which they die away depends upon the period, the quicker the rate of propagation, the faster the disturbances die away. Both these effects would be very detrimental to the distinct propagation of messages over a long wire.

Case II. When $4\pi p/a^2\sigma$ is moderately small, say between 1/10 and 4, but large enough to make $c\sigma/4\pi pa^2$ a smallish fraction. In this case by equation (2), we have

$$m = \frac{p}{V} \left\{1 - i \frac{c\sigma}{4\pi pa^2}\right\},$$

this represents a disturbance propagated with the velocity V, and fading away to $1/e$ of its value after travelling over a distance $4\pi a^2/Vc\sigma$, thus in this case both the velocity of propagation and the rate at which the vibrations die away are independent of the period, and therefore a wire fulfilling the conditions of this case will be able to transmit telephonic messages under the most favourable circumstances.

Case III. When $4\pi\mu pa^2/\sigma$ is large. In this case

$$J_0'(ina) = -J_0(ina),$$

and equation (1) becomes

$$k^2 = i \cdot \frac{p^2 2c}{V^2} \left\{\frac{\sigma\mu}{4\pi pa^2}\right\}^{\frac{1}{2}},$$

or

$$m^2 = \frac{p^2}{V^2}\left[1 + ic\left\{\frac{\sigma\mu}{\pi pa^2}\right\}^{\frac{1}{2}}\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now consider the case when $c\{\sigma\mu/\pi pa^2\}^{\frac{1}{2}}$ is small, then this equation becomes

$$m = \frac{p}{V}\left[1 + \tfrac{1}{2}ic\left\{\frac{\sigma\mu}{\pi pa^2}\right\}^{\frac{1}{2}}\right],$$

and this represents a disturbance propagated with the velocity V and fading away to $1/e$ of its value after traversing a distance

$$\frac{2V\delta}{c}\left\{\frac{\pi a^2}{\sigma\mu p}\right\}^{\frac{1}{2}}.$$

Thus in this case though the velocity of propagation is independent of the period, the rate at which the vibrations die away is not, so that if the distance telephoned over is long enough to make the disturbance die away appreciably, the message will get confused.

Case IV. When $4\pi\mu pa^2/\sigma$ is large, and $c\{\sigma\mu/\pi pa^2\}^{\frac{1}{2}}$ considerably greater than unity,

$$m = \frac{p}{V}\sqrt{c}\left\{\frac{\sigma\mu}{\pi pa^2}\right\}^{\frac{1}{4}}\left\{\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right\}.$$

In this case the rapidity of transmission is

$$Vc^{-\frac{1}{2}}\left\{\frac{\sigma\mu}{4\pi pa^2}\right\}^{-\frac{1}{4}},$$

and the distance traversed before the vibration sinks to $1/e$ of its initial value is

$$Vc^{-\frac{1}{2}}p^{-\frac{3}{4}}\left\{\frac{\sigma\mu}{4\pi a^2}\right\}^{-\frac{1}{4}};$$

in this both the velocity and rapidity of decay depend upon the period, the velocity varying less quickly with the period, and the rate of decay more quickly than Case I.

Let us now consider some numerical cases; in what follows we shall suppose the period p to range from $2\pi \times 100$ to $2\pi \times 400$; and we shall suppose that c is of the same order as if the wire

were surrounded by a conductor at zero potential at a distance 4 metres from it, or that c is of the order 1/15; this is a small estimate for c, but the effect of any change is easily calculated as the distance the vibration travels before falling to $1/e$ of its value is in Cases I. and III. inversely proportional to c, and in Cases II. and IV. to the square root of c.

The specific resistance of copper is taken as 1600, that of iron 10^4, and μ for iron 500.

Substance of which the wire is made.	Diameter of wire in centimetres.	Case under which it falls.	Distances in kilometres to which slowest and fastest vibrations travel before falling to $1/e$ of their original value.
Copper	·4	II	1500—1500
Iron	·4	IV	150—60
Copper	·2	II	400—400
Iron	2	IV	100—40
Copper	·1	I	160—80
Iron	·1	I	64—32
Copper	·02	I	32—16
Iron	·02	I	16—6

(2) *On the Effect of Pressure and Temperature on the Electric Strength of Gases.* By J. J. THOMSON, M.A., F.R.S., Cavendish Professor of Experimental Physics, Cambridge.

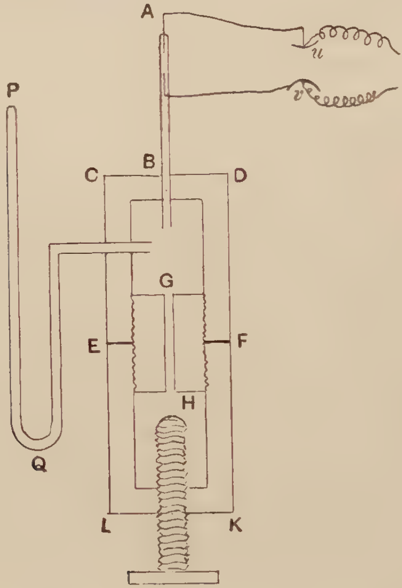
THE following experiments were undertaken to investigate the changes which take place in the electric strength of a condensible gas when the pressure is increased beyond the point necessary to liquefy it.

The gas chosen was carbonic acid, and the electric strength was tested in the following way.

CDEF is an iron rod with a cylindrical cavity bored in it, into the top of this a glass tube *AB* with very thick walls is fastened with sealing wax, and into the bottom of it an iron screw *EGF'H* with a hole *GH* running up the middle of it is screwed. *EKLF* is another iron tube filled with mercury and containing a screw *NM* which can be screwed up and down, this is screwed on to the piece *E'GF'H*. A manometer gauge *PQ* is let into *ECDF* and indicates the pressure of the gas in that vessel.

The glass tube *AB* is initially open at both ends, and CO_2 is allowed to run into *ECDF* through the glass tube and out at *H* for 6 or 7 hours. The end *A* is then fused up, and the hole at *H* closed with a small piece of paper. This portion of

the apparatus is then screwed on to $E'GF'H$, the end H dipping under the mercury, then by screwing in the screw a pressure sufficient to liquefy the CO_2 can easily be obtained; the piece of paper placed at H being pierced by the mercury when the pressure is increased. Two electrodes are fused in the tube and are placed in multiple arc with two brass balls U and V of very large radius, the distance between which can be adjusted by means of a screw. U and V are connected with the poles of an induction coil. To estimate the electric strength of the CO_2 the distance between the balls U and V is altered until as many



sparks pass through the CO_2 as through the air, when the distance between the balls U and V is taken as a measure of the electric strength of the carbonic acid. This method, though not free from objection, is very convenient, and if care be taken to keep the air surrounding the balls free from dust and carefully to prevent by continued polishing the surface of the balls from getting roughened by the sparking gives consistent results.

Experiments were made at different pressures and it was found that the electric strength of the CO_2 continually increased with the pressure and that this increase goes on past the point of liquefaction, that is, that the strength of the liquid CO_2 is greater than that of the gaseous CO_2 just before liquefaction. It was found impossible to determine more than a lower limit

for the electric strength of the liquid CO_2 for the first spark which passed by liberating gas caused such an increase in the pressure that the tube was invariably broken. The experiments however show that the liquid CO_2 possesses greater electric strength than gaseous CO_2 under great pressure, and therefore that it is a very good insulator.

The following is a specimen of the observations:

Pressure of CO_2 in atmospheres.	Air space in millimeters.
1·	6
1·25	8
2	10
3·5	11·5
6	12·25
9	14
14	15·5
23	18·

When both electrodes were surrounded by liquid CO_2 the air space was more than 24 millimetres.

Care must be taken not to allow too many sparks to pass through the CO_2, otherwise CO will be produced and liquefaction prevented.

It was found that the electric strength of highly compressed CO_2 was increased by raising the temperature, though at atmospheric pressure the electric strength of CO_2 is diminished by raising the temperature.

The electric strength of gases at ordinary pressures diminishes rapidly as the temperature increases. Thus for hydrogen at atmospheric pressure it was found that if fixed terminals in the hydrogen were placed in multiple arc with the air terminals used in the experiments before described, the distance between the terminals in the air in order that as many sparks should pass through the air as through the hydrogen had to have the following values.

Hydrogen.

Temp.	Air spaces.
15	8
120	4·5
140	3·5
155	3·5
160	3·5
175	3
200	2·5
260	2
300	2

The electric strength of gases also diminishes rapidly as the

density diminishes, in order to see whether the diminution of the strength caused by the increase of temperature could be accounted for by the rarefaction of the gas. I repeated the experiments on H . CO_2 and air, when the gases were sealed up in tubes, so that the density could not alter; from the following results it will be seen that the electric strength under these circumstances is practically constant, so that at temperatures between 15° and 300° C. the electric strength only depends on the number of molecules in unit volume.

Air at constant density.

Temp.	Air space.
10	17
20	17
35	17
50	17
60	17
70	17
80	17
100	16·5
120	16·5
140	16·5
160	16·5
180	16·5
200	16·5

Another tube.

Temp.	Air space.
18	
40	
65	
100	2
120	2·5
140	2·5
170	2·5
200	3·

CO_2 *at constant density*

Temp.	Air space.
17	2
35	1·5
65	1·5
90	1·5
140	1·5
160	1·5
185	1·5
200	2

Hydrogen at constant density.

Temp.	Air space.
13	1·5
30	1·5
50	2·5
70	2
100	2·8
120	2
140	2
160	2
180	2
200	2

(3) *On the application of Lagrange's equations to certain physical problems.* By S. H. BURBURY, M.A., St John's College.

IN his *Theory of Electricity and Magnetism*, Vol. II. chap. 7, Maxwell assumes the energy of a system of two closed currents to be of the form

$$T = \tfrac{1}{2} L_1 i_1^2 + M i_1 i_2 + \tfrac{1}{2} L_2 i_2^2,$$

where L_1, M, and L_2 are functions of the form and relative position of the circuits; and by the application of Lagrange's equations to this expression he deduces the known electromotive and electromagnetic forces.

As a logical process this reasoning is *prima facie* open to objection thus: If the reasoning be sound, it ought (it may be said) to be equally applicable to the case of a closed current and a magnetic shell. We ought to be able to express the energy of the system of circuit and shell, and to deduce the electromagnetic forces and electromotive force in the circuit by applying Lagrange's equations to that expression for the energy.

This however we cannot do. For (see Sir W. Thomson's *Papers on Electricity and Magnetism*, p. 441, note) if a closed electric current be moved in any way in the field of an invariable magnet, and the current be maintained constant, the whole energy spent, namely, the chemical energy spent in the battery, plus the mechanical work required to overcome the forces of the system, is exactly represented by an equivalent of heat generated. Therefore no change can take place in the intrinsic energy of the system, and it can have no other form than

$$T = \tfrac{1}{2} Li^2 + \tfrac{1}{2} K\phi^2,$$

where $\tfrac{1}{2} Li^2$ is the energy of the closed current i in its own field, and $\tfrac{1}{2} K\phi^2$ is the energy of the shell in its own field.

From this equation we cannot deduce the laws of induction in the circuit due to variation of the magnetic field. Maxwell's method would therefore fail if applied to the system of circuit and magnet. In order to justify it as a logical process, it becomes necessary to show how the case of circuit and shell is logically distinguishable from that of two circuits, so as to admit of the employment of Lagrange's equation in the one case and not in the other. Maxwell gives no explanation directly. It appears to me that the distinction lies in the reciprocal property mentioned by him as proved by Felici [Maxwell, § 536 (2)], namely, that the induction of circuit A on circuit X is equal to that of X on A; or as we may otherwise express it, if F_1, F_2 be the electromotive forces of induction in the two circuits respectively, $\dot{y}_1$, $\dot{y}_2$ the currents, then

$$\frac{dF_1}{d\dot{y}_2}=\frac{dF_2}{d\dot{y}_1},$$

and
$$\frac{dF_1}{d\ddot{y}_2}=\frac{dF_2}{d\ddot{y}_1}.$$

This property, though true for two circuits, is not true for circuit and shell, and that is the reason why Lagrange's equations are inapplicable in the latter case, and why Maxwell did not attempt to apply them to it.

In case of any purely mechanical system the energy is given by

$$2T=\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

over the whole system.

If $q_1 \dots q_n$ be the generalized coordinates defining the system, $p_1 \dots p_n$ the corresponding momenta, then

$$\begin{aligned}p_r=&\Sigma m\left\{\left(\frac{dx}{dq_r}\right)^2+\left(\frac{dy}{dq_r}\right)^2+\left(\frac{dz}{dq_r}\right)^2\right\}\dot{q}_r\\ &+\Sigma m\left(\frac{dx}{dq_r}\frac{dx}{dq_s}+\frac{dy}{dq_r}\frac{dy}{dq_s}+\frac{dz}{dq_r}\frac{dz}{dq_s}\right)\dot{q}_s\\ &+\&\text{c}.\end{aligned}$$

And therefore
$$\frac{dp_r}{d\dot{q}_s}=\frac{dp_s}{d\dot{q}_r},$$

and
$$p=\frac{dT}{d\dot{q}}.$$

Lagrange's equations are proved only for these values of the p's. If the space coordinates be invariable, we shall have

$$F=\frac{dp}{dt},$$

and therefore $$\frac{dF_r}{d\dot{q}_s} = \frac{dF_s}{d\dot{q}_r}, \text{ \&c.}$$

It should seem that Lagrange's equations can be applied to non-mechanical systems only when they can be shown to possess this reciprocal property. The system of two circuits was shown by Felici to possess it, Maxwell § 536 (2).

If a system does not possess this property, for instance, the system of circuit and shell, then instead of deducing the forces from the energy, we must proceed in the reverse direction, deducing the energy from the forces. If i be the current and ϕ the strength of the shell, it is found by experiment that the force required to increase i is $L\frac{di}{dt} + M\frac{d\phi}{dt}$, and the force tending to increase ϕ is $K\frac{d\phi}{dt} - M\frac{di}{dt}$, and therefore the energy T is given by

$$\begin{aligned} 2T &= i(Li + M\phi) + \phi(K\phi - Mi) \\ &= Li^2 + K\phi^2. \end{aligned}$$

February 11, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Communications were made:

(1) *On Systems of Quaternariants that are algebraically complete.* By A. R. FORSYTH, M.A., F.R.S., Trinity College.

[*Abstract.*]

THE aim of the memoir is to obtain for certain systems of quaternary quantics the respective systems of concomitants that are algebraically complete, that is, are such that every concomitant of a quantic can be expressed as an algebraical (but not necessarily nor generally an integral) function of the members of the system appertaining to that quantic. The method and the course of development are similar to those in corresponding investigations relating to ternariants; in the present case they are complicated by the presence of six (non-independent) line-variables.

It is shown that the characteristic equations satisfied by quaternariants can be reduced to twelve linear partial differential equations of the first order, which are independent of one another in form; and that these twelve can be reduced to six of them,

properly chosen and absolutely independent of one another. The leading coefficient of a quaternariant satisfies three linear partial differential equations, also of the first order; and when obtained, it uniquely determines the quaternariant by being symbolized into the umbral elements of the coefficients of the quantic,—this result being a consequence of the theorem proved that every quaternariant is expressible as an aggregate of symbolic products of factors of some of five forms.

The number of quaternariants in an algebraically complete system is $N-5$, where N is the number of coefficients in the most general form of the quantic (or of the set of simultaneous quantics) with which the system is associated. A method is indicated by which the leading coefficients of these $N-5$ quaternariants can be obtained as combinations of binariants, which belong to binary quantics derivable from the original quantic. And for the quaternariants which do not involve line-variables and which belong to unipartite quantics in point-variables, it is shown that their leading coefficients can be expressed as contravariants (and invariants) of ternary quantics derivable from the original quantic.

The general theory thus indicated is applied to obtain the special results for the following cases: (i) a quadratic: (ii) two quadratics in point-variables: (iii) a lineo-linear quantic in point- and plane-variables: (iv) a linear complex: (v) a congruence of two linear complexes: (vi) a regulus of three linear complexes: and (vii) a quadratic complex.

(2) *On the stresses in rotating Spherical Shells.* By C. Chree, M.A., King's College.

[*Abstract.*]

In a previous paper the author obtained data from which a complete solution of the problem of a rotating isotropic spherical shell was deducible, but considered in detail only the cases of a solid sphere and of an extremely thin shell. In the present paper the general solution, applicable whatever the thickness of the shell, is given explicitly.

This solution is so complicated that in its applications it is essential for clearness to select representative materials, in which there is a necessary relation between the two elastic constants of the orthodox British, or bi-constant theory. The relation mainly considered is that which on the "uniconstant" hypothesis is necessarily true, and is accepted as such by most foreign elasticians. The limiting case when Young's modulus is thrice

the torsion modulus is treated also in some detail, and the results are on the whole very similar to those given by the uniconstant hypothesis. The following remarks are however limited to the results deduced from the "uniconstant" hypothesis.

Special values—·01, ·1, ·2, $\dot{3}$, ·4, ·5, $\dot{6}$, ·8, ·9, and $1-\epsilon$ where ϵ is very small—are assigned to the ratio of the radii of the inner and outer surfaces. The results obtained for the value ·01 apply to all smaller values of the ratio greater than zero. In each case the complete analysis of the state of strain and stress over both bounding surfaces is given in tables. The increases in the mean radii and the ellipticities of the surfaces are also tabulated. The greatest values of the "tendency to rupture" on the "stress-difference" and "greatest strain" theories are deduced.

These must be treated as essentially minimum values in considering the tendency of the shell to rupture, as the author was unable to *prove*, except in the case of the solid sphere and the very thin shell, that greater values might not exist elsewhere. The fact however that on either theory the greatest values found occur at the centre of the solid sphere, and in the equatorial regions of the inner surface of shells of all degrees of thickness, renders the existence of greater values on the whole improbable.

On either theory the existence of a central cavity however small approximately doubles the tendency to rupture found in a solid sphere.

To give a clearer idea of the actual magnitude of the phenomena, the results are also tabulated for the special case of a shell possessed of the elasticity of an average piece of iron its density being 17/3, its outer radius 4000 miles, and its time of rotation 24 hours. The excess of the equatorial over the polar diameter of the outer surface increases slowly from 16 miles when the sphere is solid to 23 miles when the inner radius is 1600 miles. The excess then increases much faster as the shell grows thinner, reaching 90 miles when the inner radius is 3600 miles, and 108 miles in the very thin shell. In the solid sphere the greatest value of the maximum stress-difference is 34 tons per square inch. With the existence of a cavity however small this rises to 66 tons. It remains nearly constant till the radius of the inner surface approaches 1300 miles. It attains a maximum of nearly 87 tons per square inch when the radius of the inner surface is about 3200 miles, and in the very thin shell is about 80 tons. On the greatest strain theory the tendency to rupture is greatest in the very thin shell, in which case the conclusions of the two theories are identical. Wrought iron that can stand a traction of 30 tons per square inch, and steel that can stand 60 tons are considered exceptionally strong,

so the preceding results may be of interest to theorists on the constitution of the earth.

The paper concludes with a graphical representation of some of the more important results.

February 25, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following Communications were made:

(1) *On the relationships and geographical distribution of the Land and Fresh-Water Mollusca of the Palaearctic and Nearctic Regions.* By A. H. COOKE, M.A., King's College.

THE author stated that the boundaries of the regions concerned were, from the Molluscan point of view, identical with Wallace's regions, the Palaearctic being well defined, the Nearctic scarcely defined at all towards the south. Hence the intrusion of at least ten tropical genera into the southern United States, as compared with only one tropical genus in Southern Europe. But for the accident of the exceedingly small amount of land within the tropics in the New World, this proportion would have been much larger.

With regard to the *Land Mollusca*, no subdivision of the Palaearctic Region was possible: Great Britain and Amurland showed a very large percentage of common species.

On the other hand, the genera of the Nearctic Region fell into three main groups, (1) genera common to the whole region, (2) genera peculiar to the Pacific coast, (3) genera peculiar to the E. States. The high table-land between the Sierra Nevada and the Rocky Mountains, desert and waterless, interposed a barrier between the W. and E. genera of the same nature as the line of deserts which defined the Palaearctic region.

Group (1) was composed of very minute species (which always extend over large areas, as compared with species of larger size); 83 per cent. of these were common to N. Europe, thus affording strong support to the view which established a 'Holarctic' region. Group (2) or genera peculiar to states W. of the Sierra Nevada showed more affinity to Europe (in the proportion of 45 to 37 per cent.) than did group (3), and also showed relations to genera now occurring in China. Hence it was suggested that the possible closing of Behring's Straits, and the existence of continuous land north of the chain formed by the Kurile and the Aleutian Islands,

afforded a clue to the question of the relation between the Chinese and Californian faunas.

Further, it was held that the species (and of course, genera) common to N. America and Europe had migrated into America via E. Siberia and Alaska, i.e. from W. to E., and not from E. to W. The present distribution of *Clausilia* was instanced in illustration of this point. The relations of the genera of group (3) to Europe were then briefly discussed.

The absence of land Operculates (with exceptions easily explained) from N. America was then considered. Operculates being of tropical origin, it was unlikely that they should have entered America via Siberia. The problem was, why had they not crept northward, like so many of the Pulmonata, from the focus of their maximum development, the W. Indies? The extreme variability of the climate even of the southern States of N. America, operating against migration, was alleged as one of the possible explanations of this difficult question.

The *Fresh Water Mollusca* afforded strong evidence for a 'Holarctic' province; a very large percentage of the fresh water Pulmonata found in Europe being common to N. America. The relative migratory capacities of land and fresh-water Mollusca were then discussed, the result showing a large balance in favour of the latter, only 5 per cent. of the species of land Pulmonata, as against 26 per cent. of the fresh water species, being common to Europe and N. America.

(2) *On* Lethrus cephalotes, Rhynchites betuleti *and* Chaetocnema basalis, *three species of destructive beetles.* (Plate III.) By ARTHUR E. SHIPLEY, M.A., Christ's College.

I. *Lethrus cephalotes.*

THIS beetle has caused much damage to the vines in Bulgaria, especially in the neighbourhood of Varna, during the spring and summer of 1888. I am indebted, for the specimens from which the figures are made and for some information with regard to the habits of the beetle, to Mr A. G. Brophy, Her Majesty's Vice-Consul at Varna.

The beetle (fig. 1) belongs to the Geotrupinae, and is allied to our common Dung beetles, which it resembles in its general appearance.

The colour is a dull black, and the surface of the elytra is finely punctate, the punctures being arranged in longitudinal rows. The head is large and hexagonal, it projects far forward from under the prothorax, and is a little longer than the prothorax, measuring in an antero-posterior direction. The eyes

are small, and brown in colour, they are situated close behind the lateral angles of the head and just behind the insertion of the antennae. The margin of the head is fringed with hairs. In front of the head, the large biting mandibles project; their formidable size is well shown in fig. 1. The local name *Kara terzi* or black tailor is probably a tribute to the shear-like character of these appendages. The antennae are very characteristic, they arise just in front of the eyes, and pass downward and backward, so that their distal ends are not far from the ground. They consist of eleven joints, the last three of which are curiously modified. The first joint is large, then follow seven small joints, diminishing in length distally, then comes a large conical joint (fig. 1*a*). This ends in a flat, round disk; on this disk are two concentric circles, which are caused by the edges of the 10th and 11th joints (fig. 1*b*). These are flattened plates attached to the rim of the cone-like 9th joint at one point only, and thus they can be elevated in the way indicated in fig. 1*c*.

The prothorax is $2\frac{1}{2}$ times as broad as it is long, it is strongly arched, and its anterior border is hollowed out where the head projects. Its posterior border is almost straight. The scutellum is small.

The breadth between the shoulders is a little less than that of the prothorax, and a little longer than the whole length of the abdomen. The elytra are much arched and the posterior end of the abdomen is obtuse.

The coxae of the legs are short and stout, there are two trochanters, and the femur is provided with a row of stiff hairs borne on its inner margin. The tibia is flattened and adapted for digging, it bears a row of tubercles on its outer surface. These peculiarities are more marked in the anterior pair of legs than in the other two. The tubercles increase in size towards the distal end of the tibia, and the last one is prolonged into a thorn, longer than the first joint of the tarsus which it overlaps, opposite this is a spine of about the same length. On the posterior legs two smaller spines occupy a similar position. Along the outer side of the tibia a curious brush of closely packed hairs is situated, and four other rows of hairs run down the tibia parallel with this. The first and fifth joints of the posterior tarsi are long and each is about as long as the three middle joints taken together. Each joint bears numerous hairs, and the fifth terminates in a pair of ungues.

The length of the beetle is 2 c.m., the width of the prothorax at the broadest part 1·25 c.m.

These beetles live in pairs, in deeply tunneled holes in the ground. The male may often be seen in fine weather keeping guard at the mouth of the hole, into which it precipitately retreats

at the approach of danger. In the spring the beetles may be seen in numbers leaving their holes and hurrying to the nearest vine, up which they swarm, until they reach the youngest shoots at the end of the branches. These they proceed to cut off with their powerful mandibles, and then descending to the ground drag the succulent shoot backwards towards their holes. The shoots are left exposed to the sun for a short time, until they begin to dry up, and this is always the case after rain; when the leaves are sufficiently dried they are dragged into the holes, where Taschenberg supposes they serve as food, not so much for the beetles as for their larvae. During the heat of the day these beetles apparently rest, their visits to the vines being usually paid between 9 and 11 in the morning, and from 3 o'clock in the afternoon until sunset.

In the absence of vines, the beetle attacks and carries off to its hole the leaves of grass, dandelions and other plants. They prefer a sandy soil, and often appear near the sea-shore and spread inland. They have been known for some time as a pest in the Hungarian vineyards, but their presence in Bulgaria dates from last year. The damage is usually very local, and the vines attacked are not permanently injured, but recover the following spring. As the beetles show a marked aversion to wet weather, during which they do not leave their holes, it has been suggested that copious watering might rid a district of the pest.

II. *Rhynchites betuleti.*

I am also indebted to Mr Brophy of Varna for specimens of a very beautiful little Weevil, known as Rhynchites betuleti, which has recently caused much trouble amongst the vineyards in Bulgaria.

The family Rhynchitidae to which this beetle belongs is characterized by the antennae being straight and not elbowed (fig. 2*a*), and inserted into the middle of the proboscis, and not at the base. There is no groove on the side of the proboscis for the protection of the shaft of the antennae, and the beetles seldom exceed one-fourth of an inch in length.

Rhynchites betuleti (fig. 2) is 3 mm. broad at the base of the abdomen and 6 mm. in length from the origin of the head to the posterior extremity. It has a very metallic colour, usually green or blue, but sometimes reddish. The prothorax passes gradually into the head, no constriction or neck marking the division of these two parts. The proboscis thickens a little in front of the insertion of the antennae.

The tarsus is four-jointed, the third joint is expanded into two lappets. The elytra are thickly punctate. The male is

distinguished from the female by the possession of two thorns on the prothorax.

The habits of the beetle are as follows:—The imago creeps out of the earth in the spring, usually some time in May, the female is then fertilized and at once sets about preparing a nest for her eggs and larvae. She ascends a vine stock and carefully selecting one of the younger leaves, she bites through the mid rib, on both sides, and then folds round the lamina of the leaf into a rough roll. This is secured by a sticky secretion. Then more leaves are added until the structure assumes the appearance of a very loosely rolled cigar. In this roll of leaves four to six dirty-white eggs are laid, each about 1 mm. long. The larvae which hatch out from the eggs are yellowish-white grubs, with a brownish head and plentifully supplied with hairs. They live four or five weeks in the roll of leaves, and then descend to the ground, in which they turn to chrysalids. There is no opening in the cigar-shaped roll, as after the eggs are laid another leaf is folded over them.

The chrysalid stage lasts about a week and the whole development about sixty days. The beetles which emerge from the cocoons about the end of August pass the winter underground.

In addition to the harm caused by the female rolling up the young leaves, much damage is caused by the beetles eating the leaves. They usually prefer to attack vines which are not quite healthy. In 1872 they caused so much damage to the vineyards in the neighbourhood of Neusiedler See in Hungary, that scarcely an uninjured vine leaf was left in that district.

The only method of combating this pest is to collect and destroy the nests, which can be very easily seen.

III. *Chaetocnema basalis.*

A collection of minute beetles, together with a letter from the Deputy Conservator of Forests in the Tharrawaddy Division, Burma, describing their habits, was forwarded to me by the Inspector-General of Indian Forests, last November. The beetles were all of one species; they were preserved in spirits and were in an excellent state of preservation.

I am indebted to Mr W. F. Blandford for the identification of the species and for a careful comparison of them with the single specimen of the same species in the national collection at South Kensington. Mr Blandford identifies the beetle as *Chaetocnema basalis;* a description of a member of this species from India by Mr J. S. Baly occurs in the *Transactions of the Ento-*

mological Society of London, 1877, p. 310. The genus Chaetocnema was established by Stephens, and covers almost the same ground as the genus Plectroscelis of Chevrolat.

Mr Blandford has kindly forwarded me a description of the beetle, which I quote with his permission.

"Chaetocnema basalis (fig. 3). Short, oval, deep-black with faint greenish reflection, antennae and legs yellow, the last seven joints of the former, and the base of the anterior femora, and of the last pair pitchy, underside black. Insect winged, thorax and elytra highly polished, the former with a row of punctures at the base, the latter with regular rows of punctures.

Length ⅔ line, hab. Burma.

Head sunk in thorax, eyes rather prominent, a deep sulcus along the inner margin of each eye, these being connected in front (fig. 3*a*). The vertex of the head is very finely wrinkled and impunctate, clypeus with shallow punctures.

Thorax much broader than long, convex, with a marginal border all round, very slight along the anterior margin, deep along base which is sinuate. The anterior angles are rounded, the posterior obtuse. Disc very finely and scantily punctured, with a row of close, deep punctures along base.

Elytra short and convex, the shoulders prominent and rounded, a narrow border along suture. On the disc are eleven rows of deep punctures—counting the row along the side margin—the first row contains about ten punctures and does not reach to the middle of the elytra, it and the second row are turned slightly outwards at their base; the fourth and eighth rows meet towards the apex of the elytra, which are jointly rounded, the outer rows are obliterated on the humeral prominence, which has some fine irregular punctuation. The interstices have a row of very fine punctures, which are occasionally double.

Scutellum, small, rounded at apex, and indistinctly wrinkled.

Antennae, first two joints stout, second shorter than the first, next four much more slender and elongate, remaining joints oblong conical, the last pointed.

Tibiae, posterior pair with an obtuse spine on outer side of shaft near apex, between which and spine is a row of bristles" (fig. 3*b*).

This beetle has recently been doing serious damage to the paddy or young rice plantations in Taungyas of the northern part of the Tharrawaddy district. It appeared last year in June, when the young paddy is about 6 ins. high, and according to the Karens, who inhabit this hilly region, it eats first the leaf and then works down through the heart of the plant to the root. The Karens state that the insect appeared some years ago, but

the Burmans have no recollection of its previous occurrence: this is possibly due to the fact that it has confined its ravages to the hills, and has not descended to the plains. The native name of the insect is Wetpo.

The genus Chaetocnema belongs to the family Halticidae. This family includes a number of very minute beetles, and it is very homogeneous, the differences between the various species being very slight. A common feature of the group is the adaptation of the posterior pair of legs for leaping, a feature which, together with their minute size, has given rise to the common name Flea-beetle. One of the commonest English species is the Turnip-fly or Flea-beetle—Haltica nemorum. Much of the damage caused by these beetles is due to the larvae boring through the mesophyll of the leaves of the plant attacked, or else devouring all the soft tissue, leaving only the fibro-vascular bundles uneaten.

Amongst the methods which have been successfully used in various parts of the world in dealing with the injurious members of the Halticidae the following may be mentioned:—(i) sprinkling the affected plants with any kind of finely divided matter, innoxious to the plant, such as powdered lime, soot, road dust, ashes, sulphur, etc. This probably impedes the movements of the beetles by clogging their legs, and they always avoid plants which have been so treated; (ii) sprinkling the plants with a solution of whale-oil soap, in the proportion of 2 lbs. of soap to 16 galls. of water, or with an extract of wormwood made by pouring a pailful of boiling water over a handful of wormwood. The latter bitter extract renders the plant distasteful to the beetles.

As a rule the members of the Halticidae avoid damp places, and I understand the rice in the Taungyas is grown by the dry method of cultivation, which may account to some extent for the presence of the Chaetocnema basalis.

PROF. CAYLEY, VICE-PRESIDENT, IN THE CHAIR.

(3) *On the Skeleton of* Rhytina gigas *lately acquired for the Museum of Zoology and Comparative Anatomy; with some account of the history and extinction of the animal.* By J. W. CLARK, M.A., *President.*

THE author began by pointing out that while the living Sirenians all inhabit tropical or sub-tropical regions, the gigantic animal before them had been discovered by the German naturalist George William Steller, on the island afterwards called Bering's Island.

This island lies off the coast of Kamtschatka, between the 50th

and 60th parallels of north latitude. In shape it is an irregular triangle, of which the apex is turned towards the south-east. The length of the island is about 100 miles, and the greatest breadth about 14 miles. The seaboard, on every part of which, according to Steller, the *Rhytina* was to be found, was therefore about equal to the distance from the mouth of the Thames to the mouth of the Humber.

He then gave a detailed account of Bering's last voyage, from the origin of the expedition in 1732, to the wreck of the vessel, 5 November, 1741; after which he read a translation of Steller's account of the Manati, or Sea-cow, as he called it, from the paper drawn up from Steller's notes, and communicated to the St Petersburg Academy in 1749, three years after his death (*Nov. Comment. Acad. Scient. Imp. Pet.* ii. 289—330). This part of the communication, which dealt with the habits of the animal, as well as with the zoological characters, was illustrated by reference to the observations of those who have lately had opportunities of studying Manatees in captivity; and to the elaborate papers by J. F. Brandt, whose *Symbolæ Sirenologicæ* contain a full history of *Rhytina* and a comparison of it with *Manatus* and *Halicore*.

He then discussed Steller's measurements of the animal, taken from a female, killed 12 July, 1742, and exhibited a diagrammatic figure, drawn in accordance with his interpretation of them.

The gradual extinction of the animal was then related. On Bering's return, Russian hunters and merchants, excited by the prospect of obtaining large cargoes of skins of Arctic fox, sea-otter, and sea-lion, with which the island abounded, fitted out expeditions, sometimes two or three in each year, from 1743 to 1762. These vessels, carrying on an average a crew of about 40 men, usually wintered at Bering's Island, "in order to procure a stock of salted provisions from the sea-cows and other amphibious animals that are found there in great abundance." (*Account of the Russian Discoveries between Asia and America.* By William Coxe, M.A., F.R.S. 8vo. Lond. 1786, p. 57.) In addition to the wintering of large bodies of men on the island, small parties of hunters frequently staid there all the year round, for the purpose of trapping. These men, in their endeavours to obtain fresh meat, frequently harpooned a sea-cow, but, as they were unable to drag the carcase on shore, the animal escaped, and died in deep water. In consequence of this wasteful habit the sea-cow had become extinct on Copper Island, a small island at no great distance from Bering's Island, by 1754, when the engineer Jakoffleff, whose diary has fortunately been printed, found himself unable to winter there, on account of the scarcity of provisions, and was compelled to remove to Bering's Island. On his return he memorialised the authorities at Kamtschatka, praying them to put a stop to this

destruction on Bering's Island, "for fear the sea-cow should become extinct there also." This advice was disregarded, and Captain Billings, who explored Bering's Sea between 1785 and 1794, records: "sea-cows were very numerous... but the last of this species was killed in 1768 on Bering's Island, and none have ever been seen since." This statement is confirmed by the diary of Bragin, who wintered on Bering's Island in 1772, and enumerates the animals to be met with there, omitting *Rhytina;* and by that of Schelichoff, who wintered there in 1783, and gives similar information.

The survivors of Bering's expedition built themselves a vessel out of the wreck, with which they reached Petropaulski at the end of August, 1742. Steller had wished to bring back with him a skeleton of a young sea-cow, or at least some fragments, "spolia quædam," but the size of the craft did not admit of this; his specimens amounted to no more than two of the horny plates which were attached to the palate of the sea-cow, and did duty for teeth, as in *Manatus* and *Halicore*. These fragments, and a sketch of the animal, which appears in the *Zoographia* of Pallas, but without history or authority, constituted (with Steller's paper and diary) the whole evidence respecting *Rhytina* for more than a century. The Academy of St Petersburg offered rewards for information and specimens, but without success. They were told that "strange bones were found in the soil of Bering's Island," but failed to obtain any of them until 1844, when a broken skull was brought to Brandt. Subsequently he obtained two skeletons, one of which is nearly complete, and appears to be composed of associated bones. On these ample materials he founded the elaborate monograph contained in the *Symbolæ Sirenologicæ*.

Bering's Island is now let by the Russian Government to the Alaska Commercial Company, and by their permission, the authorities of the National Museum at Washington, U.S., have caused diligent search to be made for bones of *Rhytina* with great success. The skeleton in the Cambridge Museum was obtained from Washington, through the kindness of the late Professor Spencer F. Baird, at the instance of Professor Newton. The missing parts have been modelled from the more complete skeleton in the British Museum, by the kind permission of Dr Woodward, Director of the Geological Department.

March 11, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

The Very Rev. H. M. Butler, D.D., Master of Trinity College.
W. A. Wright, M.A., Vice-Master of Trinity College.
E. W. Gosse, M.A., Trinity College.

The President announced that Prof. Stokes, Prof. Adams, and Lord Rayleigh, the adjudicators of the Hopkins Prize for the period 1874—1876, have recommended that the Prize be awarded to SIR WILLIAM THOMSON for his Mathematical researches on the *Theory of the Tides* and other important investigations in Mathematical Physics; and that this recommendation has been confirmed by the Council of the Society.

The following Communications were made:

(1) *Remarks on a paper by* SIR G. AIRY. By F. J. CANDY, M.A., Emmanuel College.

The author gave a geometrical interpretation, by means of vectors, of the function employed by Sir G. Airy (*Proc.* Vol. VI. p. 104).

(2) *On the change of the pentad to the triad nitrogen atom.* By S. RUHEMANN, M.A., Gonville and Caius College.

(3) *On the change of citric acid to pyrrol derivatives.* By S. SKINNER, M.A., Christ's College.

May 6, 1889.

PROFESSOR ADAMS, MEMBER OF COUNCIL, IN THE CHAIR.

The following were elected Fellows of the Society:

W. H. Caldwell, M.A. Gonville and Caius College.
L. E. Shore, M.A., M.B., St John's College.
W. B. Hardy, B.A., Gonville and Caius College.

The following Communications were made:

(1) *On the binodal quartic and the graphical representation of the elliptic functions.* By Prof. CAYLEY.

[*Abstract.*]

The subject is approached from the question of the graphical representation of the elliptic functions. Starting from the equation

$x+iy = \text{sn}\,(x'+iy')$ this establishes a (1, 1) correspondence between the xy infinite quarter plane and the $x'y'$ rectangle (sides K and K'): it is shown how to the contour $A'B'C'D'$ (A' the origin, B', D' on the axes of x', y' respectively) there corresponds the contour $ABCD$ of the infinite quarter plane (A the origin, B at the distance 1 and C at the distance $\frac{1}{k}$ on the axis of x, D at infinity, that is extending from infinity on the axis of x to infinity on the axis of y): and this shows at once the general form of the curve in the xy quarter plane corresponding to any given curve in the $x'y'$ rectangle; for instance, to a straight line $E'F'$ parallel to the axis of x' and extending from E' on $A'D'$ to F' on $B'C'$ there corresponds an arc EF extending from E on AD to F on BC, and cutting each of these lines at right angles: and so in other cases. The curves thus corresponding to straight lines $E'F'$ and $G'H'$ parallel to the axes of x' and y' respectively are as is known Bicircular Quartics, and reference is made to an interesting paper by Siebeck, *Crelle*, t. 57 (1860) and t. 59 (1861): although the two theories are substantially identical, some of the properties are in the first instance developed in regard to the binodal quartic, and the term binodal quartic is accordingly introduced into the title of the Memoir. The bicircular quartics arising from the elliptic functions are biaxal curves, represented by an equation of the form

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 + C = 0.$$

(2) *A Method of discovering Particular Solutions of Certain Differential Equations, that satisfy Specified Boundary Conditions.* By J. Brill, M.A., St. John's College.

1. Riemann has given a method* by means of which we can discover solutions of a particular type of linear partial differential equations of the second order, with two independent variables, which give a specified value for the dependent variable along a given curve, as well as a specified value for its rate of variation in the direction of the normal to the curve. But in the majority of problems in Mathematical Physics, whose discussion involves linear equations of the second order, the boundary conditions are not of this character. We either have the value of the dependent variable given along each of two bounding curves, or we have its rate of variation in the direction of the normal given along each of the two curves. Or, speaking generally, instead of having two boundary conditions to satisfy along one given curve, we have one

* See a Memoir entitled "Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite." *Gesammelte Werk*, p. 145. A very lucid account of Riemann's method is given by Darboux, "*Leçons sur la Théorie Générale des Surfaces*," Livre IV., Ch. IV.

boundary condition to satisfy along each of two given curves. In what follows I have endeavoured to adapt Riemann's method to obtain solutions of problems in which the boundary conditions are of this character. I have not succeeded in making the process purely synthetical, and the portion of the work that is of a tentative character will in general be found to amount to a very difficult piece of algebra. I think, however, that this will be found to be a simpler matter than the guessing of the transcendental function that gives rise to a solution suitable to a given form of boundary. Also I am of opinion that by studying Riemann's methods of working we shall eventually be enabled to extend and to adapt them so as to render the process purely synthetical.

2. The majority of the equations that turn up are not of a form suitable for treatment by means of Riemann's method, but this difficulty can be surmounted in a large number of cases with the aid of transformation. Thus consider the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

If we write $\xi = x+iy$ and $\eta = x-iy$, the transformed equation becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta}=0,$$

which is of a form suitable for treatment by Riemann's method. If, therefore, the equations of the bounding curves when expressed in terms of ξ and η are real, and the values at the boundaries of the dependent variable, or the values of its normal variation, as the case may be, are also real; then the transformed problem will be capable of treatment by the method in question. We proceed to show how the method may be adapted to suit this particular equation.

We have the theorem

$$\iint \frac{\partial^2 u}{\partial x \partial y}\,dx\,dy = \tfrac{1}{2}\int\left(\frac{\partial u}{\partial y}\,dy-\frac{\partial u}{\partial x}\,dx\right),$$

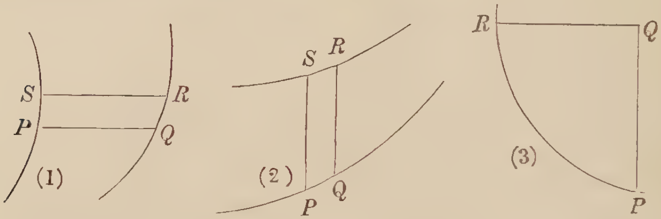
the line integral being taken round the boundary of the region throughout which the surface integral is taken. Hence, if we have

$$\frac{\partial^2 u}{\partial x \partial y}=0$$

throughout the said region, we have also

$$\int\left(\frac{\partial u}{\partial y}\,dy-\frac{\partial u}{\partial x}\,dx\right)=0,$$

the integral being taken along the boundary.



If we apply this theorem to the region $PQRS$ in figure (1), we have

$$-\int_P^Q \frac{\partial u}{\partial x}\,dx + \int_Q^R \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right)$$
$$- \int_R^S \frac{\partial u}{\partial x}\,dx + \int_S^P \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right) = 0;$$

and therefore

$$u_R - u_Q + \int_Q^R \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right) = u_S - u_P + \int_P^S \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right).$$

If we now make PQ and SR approach indefinitely near to each other, we obtain

$$\left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right)_Q + \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right)_Q$$
$$= \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right)_P + \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right)_P,$$

i.e.
$$\left(\frac{\partial u}{\partial y}\,dy\right)_Q = \left(\frac{\partial u}{\partial y}\,dy\right)_P.$$

If we apply our theorem to the region $PQRS$ in figure (2), we have

$$\int_P^Q \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right) + \int_Q^R \frac{\partial u}{\partial y}\,dy + \int_R^S \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right) + \int_S^P \frac{\partial u}{\partial y}\,dy = 0;$$

and therefore

$$u_P - u_Q + \int_P^Q \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right) = u_S - u_R + \int_S^R \left(\frac{\partial u}{\partial y}\,dy - \frac{\partial u}{\partial x}\,dx\right).$$

If we make PS and QR approach indefinitely near to each other, we obtain

$$-\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)_P+\left(\frac{\partial u}{\partial y}dy-\frac{\partial u}{\partial x}dx\right)_P$$
$$=-\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)_S+\left(\frac{\partial u}{\partial y}dy-\frac{\partial u}{\partial x}dx\right)_S,$$

i.e.
$$\left(\frac{\partial u}{\partial x}dx\right)_P=\left(\frac{\partial u}{\partial x}dx\right)_S.$$

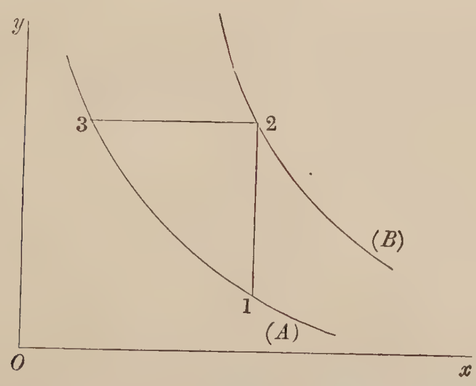
Finally, we will apply our theorem to the region PQR in figure (3). It then takes the form

$$\int_P^Q\frac{\partial u}{\partial y}dy-\int_Q^R\frac{\partial u}{\partial x}dx+\int_R^P\left(\frac{\partial u}{\partial y}dy-\frac{\partial u}{\partial x}dx\right)=0;$$

and therefore we have

$$2u_Q-(u_P+u_R)=\int_P^R\left(\frac{\partial u}{\partial y}dy-\frac{\partial u}{\partial x}dx\right).$$

3. We now proceed to show how the results of the preceding article may be utilized to obtain the solutions of particular problems.



Suppose the curves (A) and (B) in the accompanying figure to be the curves along which the specified boundary conditions are to be satisfied. On (A) take a point 1; through 1 draw 1 2 parallel to Oy to meet (B) in 2; and through 2 draw 2 3 parallel to Ox to meet (A) in 3. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) denote the co-ordinates of the points 1, 2, 3. Then by the preceding article we have

$$\frac{\partial u}{\partial x_1}dx_1=\frac{\partial u}{\partial x_2}dx_2 \text{ and } \frac{\partial u}{\partial y_2}dy_2=\frac{\partial u}{\partial y_3}dy_3.$$

In the first instance we will suppose that u is to have a constant value along each of the curves (A) and (B). Then in addition to the above equations we have the three following

$$\frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial y_1}\,dy_1 = 0,$$

$$\frac{\partial u}{\partial x_2}\,dx_2 + \frac{\partial u}{\partial y_2}\,dy_2 = 0,$$

$$\frac{\partial u}{\partial x_3}\,dx_3 + \frac{\partial u}{\partial y_3}\,dy_3 = 0.$$

Hence it follows that

$$\frac{\partial u}{\partial y_1}\,dy_1 = \frac{\partial u}{\partial y_2}\,dy_2 = \frac{\partial u}{\partial y_3}\,dy_3 = -\frac{\partial u}{\partial x_1}\,dx_1 = -\frac{\partial u}{\partial x_2}\,dx_2 = -\frac{\partial u}{\partial x_3}\,dx_3.$$

At this point a knowledge of the general functional solution of our equation will prove of service. This, which is well known, is readily deducible from the results of the preceding article. In that article we have practically proved that $\partial u/\partial x$ remains constant so long as x is unaltered, and that $\partial u/\partial y$ remains constant so long as y is unaltered. We have, therefore,

$$\frac{\partial u}{\partial x} = F'(x), \quad \frac{\partial u}{\partial y} = f'(y), \quad \text{and} \quad u = F(x) + f(y);$$

and it is evident that if u is to be real when $x+iy$ and $x-iy$ are substituted for x and y respectively, then F and f must be of the same form. Thus we shall have

$$u = f(x) + f(y),$$

and it will follow that $\partial u/\partial x$ is the same function of x as $\partial u/\partial y$ is of y.

Suppose now that

$$\phi(x,\ y) = 0 \quad \text{and} \quad \psi(x,\ y) = 0,$$

are the respective equations of the bounding curves (A) and (B). Then, since $x_1 = x_2$ and $y_2 = y_3$, we have the three equations

$$\phi(x_1,\ y_1) = 0, \quad \psi(x_1,\ y_3) = 0, \quad \phi(x_3,\ y_3) = 0;$$

and from these equations we can obtain the values of the ratios

$$dx_1 : dx_3 : dy_1 : dy_3.$$

Hence, with the aid of the above relations, we obtain the values of the ratios

$$\frac{\partial u}{\partial x_1} : \frac{\partial u}{\partial x_3} : \frac{\partial u}{\partial y_1} : \frac{\partial u}{\partial y_3}.$$

Now the value of $\partial u/\partial y$ at any point of one of the bounding curves can be expressed in terms of the y of that point, as also the value of $\partial u/\partial x$ at the said point can be expressed in terms of the x of the point. And it is obvious that $\partial u/\partial y_1$ is the same function of y_1 as $\partial u/\partial y_3$ is of y_3, and also that $\partial u/\partial x_1$ is the same function of x_1 as $\partial u/\partial x_3$ is of x_3. And we have already established that $\partial u/\partial y$ is the same function of y as $\partial u/\partial x$ is of x. What we have to do, therefore, is to express the ratios

$$dy_1 : dy_3 : dx_1 : dx_3,$$

found as above, in the form

$$\chi(y_1) : \chi(y_3) : \chi(x_1) : \chi(x_3),$$

with the aid of the three equations given above. We are then at liberty to assume that

$$\frac{\partial u}{\partial y} = \frac{k}{\chi(y)} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{k}{\chi(x)}$$

along the curve (A). It will be found that these assumptions enable us to obtain a solution. And we know that with a given form of boundary only one solution is possible; and although there are an indefinite number of solutions corresponding to boundaries including portions of the two curves as parts, yet it is reasonable to suppose that the simplest result that can be arrived at will in general apply to the case in which the boundary consists of the two complete curves, and does not involve any portion of any other curve.

Since along the curve (A) we have the relation

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0,$$

the third formula of Article 2 becomes

$$u_Q - \alpha = \int_P^R \frac{\partial u}{\partial y}\,dy = -\int_P^R \frac{\partial u}{\partial x}\,dx,$$

where α is the constant value of u along the curve (A). This formula will enable us to complete the solution as may be seen from the examples worked out below.

4. Our theorems are also very easily adapted to other forms of the boundary conditions, besides that discussed in the preceding article. Thus suppose that we have certain expressions given for the values of the dependent variable along the bounding curves, which are not constant all along those curves. Then, supposing that the functions expressing them remain real in

the transformed problem, we shall have three equations of the form

$$\frac{\partial u}{\partial x_1}\,dx_1+\frac{\partial u}{\partial y_1}\,dy_1=\frac{\partial f}{\partial x_1}\,dx_1+\frac{\partial f}{\partial y_1}\,dy_1,$$
$$\frac{\partial u}{\partial x_2}\,dx_2+\frac{\partial u}{\partial y_2}\,dy_2=\frac{\partial F}{\partial x_2}\,dx_2+\frac{\partial F}{\partial y_2}\,dy_2,$$
$$\frac{\partial u}{\partial x_3}\,dx_3+\frac{\partial u}{\partial y_3}\,dy_3=\frac{\partial f}{\partial x_3}\,dx_3+\frac{\partial f}{\partial y_3}\,dy_3.$$

The second of these equations may be written in the form

$$\frac{\partial u}{\partial x_1}\,dx_1+\frac{\partial u}{\partial y_3}\,dy_3=\frac{\partial F}{\partial x_2}\,dx_2+\frac{\partial F}{\partial y_2}\,dy_2;$$

and, subtracting this from the first equation, we obtain

$$\frac{\partial u}{\partial y_1}\,dy_1-\frac{\partial u}{\partial y_3}\,dy_3=\frac{\partial f}{\partial x_1}\,dx_1+\frac{\partial f}{\partial y_1}\,dy_1-\frac{\partial F}{\partial x_2}\,dx_2-\frac{\partial F}{\partial y_2}\,dy_2.$$

Now it is clear that $\partial f/\partial x_1$, $\partial f/\partial y_1$ and dx_1/dy_1 can be expressed in terms of y_1, and also that $\partial F/\partial x_2$, $\partial F/\partial y_2$ and dx_2/dy_2 can be expressed in terms of y_3. Consequently we shall arrive at an equation of the form

$$\left(\frac{\partial u}{\partial y_1}+\lambda\right)dy_1=\left(\frac{\partial u}{\partial y_3}+\mu\right)dy_3,$$

where λ is a function of y_1, and μ a function of y_3; and it is plain that we shall obtain similar equations connecting $\partial u/\partial x_1$ and $\partial u/\partial x_3$, $\partial u/\partial x_1$ and $\partial u/\partial y_1$, also $\partial u/\partial x_3$ and $\partial u/\partial y_3$. It is however plain that the finding of the values of $\partial u/\partial x$ and $\partial u/\partial y$, in terms of x and y respectively, from these equations, will be a much more difficult matter than it was in the case discussed in the preceding article.

The case in which the values of the rate of variation in the direction of the normal of the dependent variable are given along each of the bounding curves, is very similar to this. We shall have three equations of which the following is a type:

$$\frac{\partial u}{\partial y_1}\,dy_1-\frac{\partial u}{\partial x_1}\,dx_1=f(x_1,\,y_1)\,dy_1\left\{\left(\frac{\partial\phi}{\partial x_1}\right)^2+\left(\frac{\partial\phi}{\partial y_1}\right)^2\right\}^{\frac{1}{2}}\Big/\frac{\partial\phi}{\partial x_1}$$
$$=\theta(y_1)\,dy_1,\quad\text{say.}$$

Thus it is easily seen that we shall have equations of exactly the same type as before, viz.

$$\left(\frac{\partial u}{\partial y_1}+\lambda\right)dy_1=\left(\frac{\partial u}{\partial y_3}+\mu\right)dy_3,\ \ \&c.$$

5. As a first example of the application of these methods we will take a very simple and well-known problem. We will show how to find a function of x and y that satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

throughout the region between the curves

$$x^2+y^2=a^2 \quad \text{and} \quad x^2+y^2=b^2,$$

$(b>a)$, and has a constant value along each of the curves.

Transforming this problem, we shall have to find a value of u which satisfies the equation

$$\frac{\partial^2 u}{\partial x \partial y}=0$$

throughout the region between the curves $xy=a^2$ and $xy=b^2$, and has a constant value along each of the said curves.

Differentiating the equations

$$x_1y_1=a^2, \quad x_1y_3=b^2, \quad x_3y_3=a^2,$$

we obtain

$$\begin{aligned} x_1dy_1+y_1dx_1&=0,\\ x_1dy_3+y_3dx_1&=0,\\ x_3dy_3+y_3dx_3&=0; \end{aligned}$$

from which we deduce

$$\frac{dy_1}{y_1}=\frac{dy_3}{y_3}=-\frac{dx_1}{x_1}=-\frac{dx_3}{x_3}.$$

Therefore we have

$$y_1\frac{\partial u}{\partial y_1}=y_3\frac{\partial u}{\partial y_3}=x_1\frac{\partial u}{\partial x_1}=x_3\frac{\partial u}{\partial x_3}.$$

Thus we shall obtain a solution of our problem by assuming that

$$\frac{\partial u}{\partial y}=\frac{k}{y} \text{ and } \frac{\partial u}{\partial x}=\frac{k}{x},$$

all along the curve $xy=a^2$.

We may now complete the solution by applying the third formula of Article 2, as simplified in Article 3. We will use (x, y) for the coordinates of Q, taken as any point between the two bounding curves, and (x', y') for the current coordinates along the curve PR. We will also take α as the constant value of u along the curve $xy=a^2$, and β that along the curve $xy=b^2$. Then we have

$$u_Q-\alpha=k\int_{\frac{a^2}{x}}^{y}\frac{dy'}{y'}=k\left\{\log y-\log\frac{a^2}{x}\right\}=k\log\frac{xy}{a^2}.$$

Therefore $$\beta - \alpha = k \log \frac{b^2}{a^2},$$

and we obtain

$$u_Q = \alpha + (\beta - \alpha) \log \frac{xy}{a^2} \Big/ \log \frac{b^2}{a^2}.$$

Thus the solution of the original problem is

$$u = \alpha + (\beta - \alpha) \log \frac{x^2 + y^2}{a^2} \Big/ \log \frac{b^2}{a^2}.$$

6. We will next take as our bounding curves the two circles

$$x^2 + y^2 - 2hx + \delta^2 = 0,$$

and $$x^2 + y^2 - 2kx + \delta^2 = 0.$$

In the transformed problem the equations of the bounding curves will be

$$xy - h(x + y) + \delta^2 = 0,$$

and $$xy - k(x + y) + \delta^2 = 0.$$

Thus we shall have the three equations

$$x_1y_1 - h(x_1 + y_1) + \delta^2 = 0,$$
$$x_1y_3 - k(x_1 + y_3) + \delta^2 = 0,$$
$$x_3y_3 - h(x_3 + y_3) + \delta^2 = 0.$$

Eliminating x_1 from the first two of these equations we obtain

$$(h - k)(y_1y_3 - \delta^2) - (hk - \delta^2)(y_1 - y_3) = 0.$$

Differentiating this we have

$$dy_1 \{(h - k) y_3 - (hk - \delta^2)\} + dy_3 \{(h - k) y_1 + hk - \delta^2\} = 0;$$

and therefore

$$dy_1 \{y_3 (y_1 - y_3) - (y_1y_3 - \delta^2)\} + dy_3 \{y_1 (y_1 - y_3) + y_1y_3 - \delta^2\} = 0;$$

i.e. $$dy_1 (y_3^2 - \delta^2) = dy_3 (y_1^2 - \delta^2).$$

In a similar manner we can obtain from the second and third equations $$dx_1 (x_3^2 - \delta^2) = dx_3 (x_1^2 - \delta^2).$$

Further, from the first equation we have

$$y_1 = \frac{hx_1 - \delta^2}{x_1 - h},$$

and therefore

$$dy_1 = \frac{h(x_1 - h) - (hx_1 - \delta^2)}{(x_1 - h)^2} dx_1 = \frac{\delta^2 - h^2}{(x_1 - h)^2} dx_1.$$

Also $$y_1^2 - \delta^2 = \frac{(hx_1 - \delta^2)^2 - \delta^2 (x_1 - h)^2}{(x_1 - h)^2} = \frac{(h^2 - \delta^2)(x_1^2 - \delta^2)}{(x_1 - h)^2};$$
and thus it follows that
$$\frac{dy_1}{y_1^2 - \delta^2} = -\frac{dx_1}{x_1^2 - \delta^2}.$$

Thus we have the relations
$$\frac{dy_1}{y_1^2 - \delta^2} = \frac{dy_3}{y_3^2 - \delta^2} = -\frac{dx_1}{x_1^2 - \delta^2} = -\frac{dx_3}{x_3^2 - \delta^2};$$
and, when combined with the equations of Article 3, these give
$$(y_1^2 - \delta^2)\frac{\partial u}{\partial y_1} = (y_3^2 - \delta^2)\frac{\partial u}{\partial y_3} = (x_1^2 - \delta^2)\frac{\partial u}{\partial x_1} = (x_3^2 - \delta^2)\frac{\partial u}{\partial x_3}.$$
We are therefore at liberty to assume that
$$\frac{\partial u}{\partial y} = \frac{\lambda}{y^2 - \delta^2} \text{ and } \frac{\partial u}{\partial x} = \frac{\lambda}{x^2 - \delta^2},$$
all along the curve
$$xy - h(x + y) + \delta^2 = 0.$$

We therefore obtain
$$u_Q - \alpha = \int_{\frac{hx - \delta^2}{x - h}}^{y} \frac{dy'}{y'^2 - \delta^2}$$
$$= \frac{\lambda}{2\delta}\left\{\log\frac{y - \delta}{y + \delta} - \log\frac{(x + \delta)(h - \delta)}{(x - \delta)(h + \delta)}\right\}$$
$$= \frac{\lambda}{2\delta}\left\{\log\frac{xy - \delta(x + y) + \delta^2}{xy + \delta(x + y) + \delta^2} - \log\frac{h - \delta}{h + \delta}\right\};$$
and consequently
$$\beta - \alpha = \frac{\lambda}{2\delta}\left\{\log\frac{k - \delta}{k + \delta} - \log\frac{h - \delta}{h + \delta}\right\}.$$

These equations determine the solution of the problem, and we have for the solution of the original problem
$$u = \alpha + (\beta - \alpha)\,\frac{\log\dfrac{(x - \delta)^2 + y^2}{(x + \delta)^2 + y^2} - \log\dfrac{h - \delta}{h + \delta}}{\log\dfrac{(k - \delta)(h + \delta)}{(k + \delta)(h - \delta)}}.$$

7. In the cases worked out in Articles 5 and 6, the expressions involved are made to fall very easily into the required forms. This

happens because $\partial u/\partial y$ and $\partial u/\partial x$ are rational functions of y and x respectively; but in cases where irrational functions turn up, it will be found that the work is in general much more difficult. We have chosen these two cases because they work out quite comfortably to the very end, and therefore serve as excellent examples of the complete process. We will illustrate the cases in which irrational functions turn up with the aid of some new examples, which we will not trouble to work beyond the point at which we find the expressions for $\partial u/\partial y$ and $\partial u/\partial x$ along one of the bounding curves.

We will now work out the case in which the boundaries consist of the axis of y and the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

In the transformed problem the equations of the bounding curves will be $x+y=0,$

and $$(x^2+y^2)(a^2-b^2)-2xy(a^2+b^2)+4a^2b^2=0;$$

and we therefore have the three equations

$$x_1+y_1=0,$$

$$(x_1^2+y_3^2)(a^2-b^2)-2x_1y_3(a^2+b^2)+4a^2b^2=0,$$

$$x_3+y_3=0.$$

If we eliminate x_1 from the first two of these equations, we obtain

$$(y_1^2+y_3^2)(a^2-b^2)+2y_1y_3(a^2+b^2)+4a^2b^2=0;$$

and differentiating this we have

$$\{y_1(a^2-b^2)+y_3(a^2+b^2)\}\,dy_1+\{y_1(a^2+b^2)+y_3(a^2-b^2)\}\,dy_3=0.$$

And if we now write

$$v=\frac{y_3-y_1}{y_1(a^2-b^2)+y_3(a^2+b^2)},\quad w=\frac{y_1-y_3}{y_1(a^2+b^2)+y_3(a^2-b^2)},$$

this equation becomes $dy_1/v=dy_3/w.$

If we now eliminate y_3 between the equation giving v and that connecting y_1 with y_3, also y_1 between the equation giving w and that connecting y_1 with y_3, we obtain the two equations

$$v^2\{(a^2+b^2)^2-y_1^2(a^2-b^2)\}-2v(y_1^2+a^2+b^2)+\frac{1}{b^2}(y_1^2+b^2)=0,$$

$$w^2\{(a^2+b^2)^2-y_3^2(a^2-b^2)\}-2w(y_3^2+a^2+b^2)+\frac{1}{b^2}(y_3^2+b^2)=0.$$

Solving these equations we obtain

$$v = \frac{b(y_1^2+a^2+b^2) \pm ay_1\{y_1^2-(a^2-b^2)\}^{\frac{1}{2}}}{b\{(a^2-b^2)^2-y_1^2(a^2+b^2)\}},$$

$$w = \frac{b(y_3^2+a^2+b^2) \pm ay_3\{y_3^2-(a^2-b^2)\}^{\frac{1}{2}}}{b\{(a^2-b^2)^2-y_3^2(a^2+b^2)\}}.$$

Thus we have, if we adopt the upper sign, the relations

$$\frac{dy_1\{(a^2-b^2)^2-y_1^2(a^2+b^2)\}}{b(y_1^2+a^2+b^2)+ay_1\{y_1^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= \frac{dy_3\{(a^2-b^2)^2-y_3^2(a^2+b^2)\}}{b(y_3^2+a^2+b^2)+ay_3\{y_3^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= -\frac{dx_1\{(a^2-b^2)^2-x_1^2(a^2+b^2)\}}{b(x_1^2+a^2+b^2)-ax_1\{x_1^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= -\frac{dx_3\{(a^2-b^2)^2-x_3^2(a^2+b^2)\}}{b(x_3^2+a^2+b^2)-ax_3\{x_3^2-(a^2-b^2)\}^{\frac{1}{2}}},$$

the latter two expressions being deduced from the former two with the aid of the equations $x_1+y_1=0$ and $x_3+y_3=0$.

In a similar manner, starting with the second and third of our original equations, we should obtain

$$\frac{dx_1\{(a^2-b^2)^2-x_1^2(a^2+b^2)\}}{b(x_1^2+a^2+b^2)+ax_1\{x_1^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= \frac{dx_3\{(a^2-b^2)^2-x_3^2(a^2+b^2)\}}{b(x_3^2+a^2+b^2)+ax_3\{x_3^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= -\frac{dy_1\{(a^2-b^2)^2-y_1^2(a^2+b^2)\}}{b(y_1^2+a^2+b^2)-ay_1\{y_1^2-(a^2-b^2)\}^{\frac{1}{2}}}$$
$$= -\frac{dy_3\{(a^2-b^2)^2-y_3^2(a^2+b^2)\}}{b(y_3^2+a^2+b^2)-ay_3\{y_3^2-(a^2-b^2)\}^{\frac{1}{2}}}.$$

If we multiply these two sets of relations together, and take the square root of the result, we have

$$\frac{dy_1\{(a^2-b^2)^2-y_1^2(a^2+b^2)\}}{[b^2(y_1^2+a^2+b^2)^2-a^2y_1^2\{y_1^2-(a^2-b^2)\}]^{\frac{1}{2}}}$$
$$= \frac{dy_3\{(a^2-b^2)^2-y_3^2(a^2+b^2)\}}{[b^2(y_3^2+a^2+b^2)^2-a^2y_3^2\{y_3^2-(a^2-b^2)\}]^{\frac{1}{2}}}$$

$$= -\frac{dx_1\{(a^2-b^2)^2 - x_1^2(a^2+b^2)\}}{[b^2(x_1^2+a^2+b^2)^2 + a^2x_1^2\{x_1^2-(a^2-b^2)\}]^{\frac{1}{2}}}$$

$$= -\frac{dx_3\{(a^2-b^2)^2 - x_3^2(a^2-b^2)\}}{[b^2(x_3^2+a^2+b^2)^2 - a^2x_3^2\{x_3^2-(a^2-b^2)\}]^{\frac{1}{2}}}\ {}^{*}.$$

We may now assume

$$\frac{\partial u}{\partial y} = \frac{\lambda\{(a^2-b^2)^2 - y^2(a^2+b^2)\}}{[b^2(y^2+a^2+b^2)^2 - a^2y^2\{y^2-(a^2-b^2)\}]^{\frac{1}{2}}},$$

and
$$\frac{\partial u}{\partial x} = \frac{\lambda\{(a^2-b^2)^2 - x^2(a^2+b^2)\}}{[b^2(x^2+a^2+b^2)^2 - a^2x^2\{x^2-(a^2-b^2)\}]^{\frac{1}{2}}}$$

all along the curve $x + y = 0$, and then the solution may be completed in the usual manner.

8. As a last instance we will take for our boundaries the curves whose equations are

$$x^2 + y^2 = c^2,$$

and
$$(x^2+y^2)^2 + c^4 = 2h^2(x^2-y^2),$$

where $h > c$.

In the transformed problem the equations of the bounding curves will be

$$xy = c^2$$

and
$$x^2y^2 + c^4 = h^2(x^2+y^2),$$

and we shall consequently have the three equations

$$x_1y_1 = c^2,$$
$$x_1^2y_3^2 + c^4 = h^2(x_1^2+y_3^2),$$
$$x_3y_3 = c^2.$$

Eliminating x_1 from the first two of these equations, we obtain

$$y_1^2y_3^2 - a^2(y_1^2+y_3^2) + c^4 = 0,$$

where $a^2 = c^4/h^2$, and differentiating this we deduce

$$y_1dy_1(y_3^2-a^2) + y_3dy_3(y_1^2-a^2) = 0.$$

If we now write

$$v = (y_1-y_3)(y_1^2-a^2)$$

and
$$w = (y_3-y_1)(y_3^2-a^2),$$

* The choice of signs in these expressions can easily be verified by transforming the former two into the latter two with the aid of the equations $x_1+y_1=0$ and $x_3+y_3=0$.

this equation becomes $y_1 dy_1/v = y_3 dy_3/w$. And by combining the equations defining v and w with the equation connecting y_1 with y_3, we easily deduce

$$v = y_1(y_1^2-a^2) \pm \{(a^2y_1^2-c^4)(y_1^2-a^2)\}^{\frac{1}{2}}$$

and

$$w = y_3(y_3^2-a^2) \pm \{(a^2y_3^2-c^4)(y_3^2-a^2)\}^{\frac{1}{2}}.$$

If we adopt the upper sign in the expressions for v and w, we obtain

$$\frac{y_1 dy_1}{y_1(y_1^2-a^2)+\{(a^2y_1^2-c^4)(y_1^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{y_3 dy_3}{y_3(y_3^2-a^2)+\{(a^2y_3^2-c^4)(y_3^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{c^2 dx_1}{a^2x_1^2-c^4-x_1\{(a^2x_1^2-c^4)(x_1^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{c^2 dx_3}{a^2x_3^2-c^4-x_3\{(a^2x_3^2-c^4)(x_3^2-a^2)\}^{\frac{1}{2}}}.$$

Similarly if we started from the second and third of our original equations we should obtain

$$\frac{x_1 dx_1}{x_1(x_1^2-a^2)+\{(a^2x_1^2-c^4)(x_1^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{x_3 dx_3}{x_3(x_3^2-a^2)+\{(a^2x_3^2-c^4)(x_3^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{c^2 dy_1}{a^2y_1^2-c^4-y_1\{(a^2y_1^2-c^4)(y_1^2-a^2)\}^{\frac{1}{2}}}$$
$$= \frac{c^2 dy_3}{a^2y_3^2-c^4-y_3\{(a^2y_3^2-c^4)(y_3^2-a^2)\}^{\frac{1}{2}}}.$$

Multiplying these two sets of relations together, and taking the square root of the result, we obtain

$$\frac{y_1^{\frac{1}{2}} dy_1}{\{y_1^2(2a^2-y_1^2)-c^4\}^{\frac{1}{2}}\{(a^2y_1^2-c^4)(y_1^2-a^2)\}^{\frac{1}{4}}}$$
$$= \frac{y_3^{\frac{1}{2}} dy_3}{\{y_3^2(2a^2-y_3^2)-c^4\}^{\frac{1}{2}}\{(a^2y_3^2-c^4)(y_3^2-a^2)\}^{\frac{1}{4}}}$$
$$= -\frac{x_1^{\frac{1}{2}} dx_1}{\{x_1^2(2a^2-x_1^2)-c^4\}^{\frac{1}{2}}\{(a^2x_1^2-c^4)(x_1^2-a^2)\}^{\frac{1}{4}}}$$
$$= -\frac{x_3^{\frac{1}{2}} dx_3}{\{x_3^2(2a^2-x_3^2)-c^4\}^{\frac{1}{2}}\{(a^2x_3^2-c^4)(x_3^2-a^2)\}^{\frac{1}{4}}} *.$$

* In this case the choice of signs may be verified by a similar method to that applied to the case in the preceding Article.

We may now assume

$$\frac{\partial u}{\partial y} = \frac{\lambda y^{\frac{1}{2}}}{\{y^2(2a^2 - y^2) - c^4\}^{\frac{1}{2}}\{(a^2y^2 - c^4)(y^2 - a^2)\}^{\frac{1}{4}}}$$

and

$$\frac{\partial u}{\partial x} = \frac{\lambda x^{\frac{1}{2}}}{\{x^2(2a^2 - x^2) - c^4\}^{\frac{1}{2}}\{(a^2x^2 - c^4)(x^2 - a^2)\}^{\frac{1}{4}}},$$

and then we may complete the solution in the usual manner.

9. In the last two cases, in accordance with a remark made in Article 3, it is evident that we cannot tell *a priori* whether we have hit upon the solution we are in search of, or whether we have hit upon one referring to a more complicated form of boundary containing portions of the given curves as parts. This matter can only be tested by working out the solution to the end, although it is not *a priori* improbable that the simplest solution obtainable may be the one required. Our difficulties are also increased by the fact that, as our original transformation is an imaginary one, there is not a real point for point correspondence between the two diagrams. However, in the case worked out in Article 5, to every circle whose radius lies in magnitude between a and b there corresponds a rectangular hyperbola lying between the curves $xy = a^2$ and $xy = b^2$; and consequently the region between the two rectangular hyperbolas may be said to correspond in a certain sense to the region between the two circles. A similar remark will apply to the case worked out in Article 6; and it is not improbable that a similar statement will apply to every case.

May 20, 1889.

MR J. W. CLARK, PRESIDENT, IN THE CHAIR.

A. C. Dixon, B.A., Trinity College,

was elected a Fellow of the Society.

The PRESIDENT announced that Prof. Sir G. G. Stokes, Bart., Sir William Thomson, and Lord Rayleigh, the adjudicators of the Hopkins Prize for the period 1877—1879, have awarded the prize to Professor G. H. DARWIN, F.R.S., for his researches on the Physics of the Earth.

The following Communications were made:

(1) *On the change of shape in turgescent pith.* By Miss A. BATESON, Newnham College, and F. DARWIN, M.A., Christ's College.

The experiments were as follows. Small blocks, cut from the turgescent pith of various plants, were placed in water, and the

changes occurring in their transverse diameter were noted. The chief result of interest is that in most cases (e.g. *Helianthus annuus*) the diameter increases at first and subsequently diminishes. In the case of *Impatiens sultani* the increase is not followed by any diminution in thickness. The results are of interest as showing the different extensibility of cell-walls in different directions.

(2) *On the thickening of the stem in various species of* Thunbergia. By M. C. Potter, M.A., Peterhouse, and W. Gardiner, M.A., Clare College.

The stems of various species of *Thunbergia* are considerably thickened just above the insertion of the leaves, these thickenings possessing a structure resembling that of the pulvinus of *Mimosa*; just as the leaves of this plant are raised or lowered by their pulvini, so the stems of *Thunbergia* are capable of motion at the nodes. The authors called attention to the climbing habits of the species of *Thunbergia*, and to the advantages gained by possessing stem-pulvini.

The observations were made simultaneously by Mr Potter on *T. laurifolia* in Ceylon, and by Mr Gardiner on *T. natalensis* in Cambridge.

(3) *On a new species of* Dinophilus. By S. F. Harmer, M.A., King's College.

This species of *Dinophilus* was found at Plymouth, in rock-pools not far below high-water mark, in March and April of the present year. The maximum length is about 2 mm., and the male and female do not differ from one another in size or in external appearance. The head bears two prae-oral rings of cilia, but has no ciliated pits; the body consists of five segments (specially distinct in young individuals) and a tail, and is covered ventrally by a uniform coating of cilia. The dorsal and lateral portions of each segment are encircled by *two* ciliated rings, as in *Protodrilus Leuckartii*, described by Hatschek. In preserved specimens, the two ciliated rings of each segment give rise to the appearance of a broad band encircling the middle region of the segment, and in allusion to this character the name *D. taeniatus* was suggested for the species. An eleventh post-oral ciliated ring occurs immediately in front of the anus.

Five pairs of nephridia were described in the female; in the male, the fifth nephridium of each side is converted into a large vesicula seminalis, into which ripe spermatozoa are received by means of a ciliated funnel. Each vesicula seminalis opens by a short duct into a median penis, which has an external aperture on the ventral side, nearly opposite the anus. Fertilization is

internal, and is effected by the thrusting of the penis through any part of the skin of the female.

The second ciliated ring of the head is composed of three rows of cilia, and appears to be the homologue of the prae-oral band of a Trochosphere. The nephridia which constitute the first pair correspond in position and in structure to the head-kidneys of a Trochosphere.

(4) *On the Formation of Struvite by Micro-organisms.* By H. Robinson, M.A., Downing College.

I wish to call the attention of the Society to the formation of Struvite, or ammonium-magnesium phosphate, or—as it is sometimes called—triple-phosphate by micro-organisms when they are cultivated in nutrient gelatin and agar-agar. Last November I noticed, for the first time, in several tubes of nutrient gelatin in which pure cultivations of micro-organisms had been growing for a long time, fine, bright, well-formed crystals. I could not get the micro-organism producing the finest crystals to grow in another tube; but a second, the Bacterium putidum fluorescens, when transferred to a tube of agar-agar grew well and produced crystals in abundance and very rapidly. This specimen I obtained from a Cambridge well water in February, 1888, and it had been grown in tube after tube from that time. I had at the same time another specimen of the same bacterium which Mr Adami had given me; so to make sure the crystals were really produced by the micro-organisms and not by any spontaneous change of the nutrient substance I determined to make some experiments with both specimens, and in two lots of agar-agar rather differently prepared. I therefore inoculated four tubes with one specimen and four with the other; in each set of experiments one tube was prepared from agar-agar, infusion of beef, peptone and salt in the usual way, and the three others with agar-agar, Liebig's extract, peptone and salt. In three days all the four tubes inoculated with my own bacterium showed a good crop of crystals; but it was not until the ninth day Mr Adami's bacterium succeeded in producing any, and then only in one tube—an extract of meat tube. On the nineteenth day two more tubes—the one made with infusion of beef, and another meat extract tube—contained crystals, but it was a week longer before I noticed any in the last of the four tubes. At this time Mr Adami's bacterium grew much more luxuriantly and produced more green colouring matter than my own; but in more recent cultivations of the two specimens I have not found this difference between them, or in their power to produce crystals.

I have isolated another micro-organism, I think a bacillus—spore-forming—in which also the ability to cause the formation

of crystals seems to be increased by cultivation; it at first produced a green colour rapidly and crystals slowly, now the order is reversed. These observations may indicate that the micro-organisms first of all secrete a ferment which decomposes the nutrient substance, so that they are therefore only indirectly concerned in producing the ammonia which is necessary for the formation of the crystals; but on the other hand certain micro-organisms which Dr Cunningham has recently obtained for me from the human mouth behave, when cultivated on agar-agar, so as to lead me to conclude their action is direct. They have produced crystals on the second day after inoculation, and these crystals have been formed at the points of most active growth, that is immediately beneath the strongest colonies. In many of my older cultivations they are also formed in this position, while in others they are irregularly distributed through the agar-agar. Whether the crystals are produced directly by the micro-organisms, or indirectly by a ferment which is the product of their lives, it is a fact that they do appear sooner or later in nearly every one of the numerous tubes in which I have a micro-organism growing, and that they never are formed in any tubes free from contamination, and I have some of the latter in my possession which I have had under observation for more than fifteen months. I have also found crystals in tubes of agar-agar and nutrient gelatin where the only growth was a mould, and in others where it was a yeast. I have analysed the crystals and have found them to consist of the double ammonium-magnesium phosphate; further, Mr Solly, of the mineralogical department, has kindly examined them for me, and has found their crystalline form identical with that of the mineral Struvite. My theory is that the micro-organisms produce the ammonia from the nitrogenous organic matter in which they are growing, and that it then combines with the magnesium phosphate present both in the nutrient gelatin and in the agar-agar to form the double salt. This formation of ammonium-magnesium phosphate in the tubes by micro-organisms appears to explain the formation of the mineral Struvite in nature. The latter has been found in old graveyards, under the floors of stables, and in guano, in exactly the places where organic matter is undergoing decomposition in the presence of magnesia and phosphoric acid. Moreover it is probable that the decomposition of organic matter only takes place through the intervention of low forms of life, for it is well known that when substances so very prone to undergo decomposition as meat, jelly, milk, urine, &c. have been sterilised by heating, they may be kept exposed to the atmosphere for almost indefinite periods without putrification taking place if the vessels containing them are plugged by cotton wool so that the air is filtered, and so is freed from micro-organisms

before entering them. It seems probable that Struvite may be formed in ordinary soils through the agency of low forms of life; if this is a fact it may, to a certain extent at least, account for the "fixation of ammonia in the soil," as the highly crystalline Struvite is a very insoluble compound.

Perhaps I may be allowed here to suggest that if the soil in Fairy rings is carefully examined Struvite will most likely be found in it.

I have also been led to think the formation of Calculi in the body may arise from the production in it of ammonia by micro-organisms. I have already stated that certain micro-organisms found in the human mouth produce these crystals with great facility. This fact has led Dr Cunningham and myself to suppose they may be the cause of the deposition of tartar on the teeth. Analyses of the saliva show the presence in it of considerable quantities of magnesium and calcium phosphates in solution. Ammonia is only wanted to cause the formation of ammonium-magnesium phosphate, and to throw out of solution the calcium phosphate, and these salts are both constituents of Tartar, and we have shown that certain micro-organisms commonly found in the mouth do produce that substance. We offer the suggestion that tartar may be produced in this way, but frankly acknowledge that further experiments must be made before we can declare that such is the case.

Since the above was written I have found that if a little dilute solution of ammonia is introduced beneath the surface of the agar-agar in a tube, small but well-shaped crystals of Struvite are formed in a few minutes.

(4) Mr W. Gardiner exhibited specimens of the leaf and inflorescence of *Aciphylla squarrosa*, presented to the Botanical Museum by Messrs Veitch.

June 3, 1889.

Mr J. W. Clark, President, in the Chair.

The following Communications were made:

(1) *Note on the Determination of Arbitrary Constants which appear as Multipliers of Semi-convergent Series.* By Professor Sir G. G. Stokes.

In three papers communicated at different times to the Society*, I have considered the application of divergent series to the actual and easy calculation, to an amply sufficient degree of accuracy, of certain functions which occur in physical investigations,

* *Camb. Phil. Trans.* Vol. ix., p. 166; Vol. x., p. 105; Vol. xi., p. 412.

but which can of course be considered quite apart from their applications. These functions present themselves as the complete integrals of certain linear differential equations, or it may be as definite integrals which lead to such differential equations, of which they form particular integrals; and as of course the theory of the complete integrals includes that of any particular integrals, the subject is best regarded from the former, and more general, point of view.

The independent variable was taken as in general a mixed imaginary, and the complete integral was expressed in two ways, either by ascending series which were always convergent, or by exponentials multiplied by descending series which were always divergent (except in very special cases in which they might terminate), though when the divergent series were practically useful they were of the kind that has been called semi-convergent. In either form of the complete integral, the arbitrary constants appeared as multipliers of the infinite series (of the ascending or descending as the case might be), or it might be, in part, of a function in finite terms. The determination of the arbitrary constants, a thing in general so easy, formed here one of the chief difficulties; and the capital problem may be stated to be, to find the relations (linear relations of course) between the arbitrary constants in the one and those in the other of these two forms of the complete integral.

In the papers referred to, this was always effected by means of a third form of the complete integral, in which it was expressed by definite integrals, their coefficients forming a third set of arbitrary constants. The first two forms of integral were useful for numerical calculation, the one or the other being preferred according as the modulus of the independent variable was small or large; the second form indeed could be used *only* when the modulus was sufficiently large, so that the adoption of the first form in that case was not *merely* a matter of preference; the first form could theoretically be used in any case, but the numerical calculation would become inconveniently or even impracticably long if the modulus were large. The third form was not convenient for numerical calculation, and was used only as a journeyman solution, for connecting the arbitrary constants in the first and second forms of integral with one another, by connecting them each in the first instance with the set in the third form of solution. I remarked that in the event of our not being able to obtain a solution of the differential equation in the form of definite integrals, the use of the first two forms of integral would not therefore fall to the ground; the linear relations between the arbitrary constants in the first and those in the second form could still be obtained numerically, though in an inelegant and more

laborious manner, by calculating numerically from the ascending and descending series for the same value of the variable, and equating the results.

My attention has recently been recalled to the subject, and I have been led to perceive that the constants in the first two forms of the integral may readily be connected without going behind the series themselves, so that the expression of the integral of the differential equation by means of definite integrals may be dispensed with altogether; and even if we failed to obtain a solution in this form the two sets of arbitrary constants could be connected exactly by means of known transcendents, and not merely approximately by numerical calculation.

The ascending, and always convergent, series treated of in the three papers already referred to were particular cases of one which, on dividing the whole by a certain power of the variable, has for general or $(m+1)$th term

$$u_m = \frac{\Gamma(m+a)\,\Gamma(m+b)\ldots}{\Gamma(m+h)\,\Gamma(m+k)\ldots}\,x^m \ldots\ldots\ldots\ldots\ldots\text{(A)},$$

there being at least one more Γ-function in the denominator than in the numerator, so that the series is always convergent. The connexion of the constants in the ascending and descending series was made to depend on two things; one, the determination of the critical amplitudes of the imaginary variable x, or $\rho(\cos\theta + i\sin\theta)$, in crossing which the arbitrary constant multiplying one of the divergent series was liable to change, and the mode of that change; the other, the determination, for some one value of θ lying within those limits, of that function of ρ to which the whole expression by ascending series was ultimately equal when ρ became infinite. The value of θ always chosen was such as to make all the terms in one of the ascending series regularly positive; accordingly in the series whose general term is written above it would be $\theta = 0$, giving $x = \rho$. Now when ρ is very large the series diverges for a great number of terms, but at last we arrive at the greatest term, u_{m_1}, suppose, after which the series begins to converge. For a great number of terms in the neighbourhood of u_{m_1}, the ratio of consecutive terms is very nearly a ratio of equality, but the product of those ratios presently begins to tell. Let α and β be two positive quantities as small as we please; then the number of integers lying between $(1-\alpha)\,m_1$ and $(1+\beta)\,m_1$ will increase indefinitely as ρ and consequently m_1 increases indefinitely, and moreover the ratio of Σu_m taken for values of m lying between the limits $(1-\alpha)\,m_1$ and $(1+\beta)\,m_1$ will ultimately bear to the whole series from 0 to ∞ a ratio of equality.

Hence in considering the ultimate value of the series we may restrict ourselves to the portion of it mentioned above.

Now when m is very great we have ultimately by a known theorem

$$\Gamma(m+a) = \sqrt{2\pi(m+a-1)}\,\{(m+a-1)/e\}^{m+a-1}$$

or

$$\sqrt{2\pi m}\,.\,m^{m+a-1}\left(1+\frac{a-1}{m}\right)^m e^{-(m+a-1)},$$

or

$$\sqrt{2\pi m}\,.\,m^{a-1}\,m^m e^{-m}.$$

Let $h+k+\ldots-a-b-\ldots=s$, and let t be the excess of the number of Γ-functions in the denominator of u_m over the number in the numerator; then the expression for u_m becomes ultimately

$$(2\pi m)^{-t/2}\,.\,m^{-s+t}\,.\,(e/m)^{tm}\rho^m \,\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

The ratio of consecutive terms, which may be obtained from this expression, or more readily directly from (A), is since m is supposed very large $m^{-t}\rho$, and hence for the greatest term we may take

$$m_1{}^t = \rho \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

Strictly speaking m_1 would be the integer next over the (in general) fractional value of m_1 which satisfies the above equation, but it is easy to see that in passing to the limit we may suppose the equation satisfied exactly. Within the specified limits of that portion of our series which it suffices to consider, we see at once that m_1 may be written for m when we are dealing with any finite power of m, since α and β may be supposed to vanish *after* ρ has been made infinite. We need therefore only attend to the last portion (v) of u_m where

$$v = (e/m)^{tm}\rho^m = e^{tm}m^{-tm}m_1{}^{tm}.$$

Now treating m as continuous, i.e. not necessarily integral, and putting w for $\log v$, we have

$$w = t(1+\log m_1 - \log m)\,m, \quad = tm_1 \text{ when } m = m_1,$$

$$\frac{dw}{dm} = t(\log m_1 - \log m), \quad = 0 \text{ when } m = m_1,$$

$$\frac{d^2w}{dm^2} = -\frac{t}{m}, \quad = -\frac{t}{m_1} \text{ when } m = m_1,$$

whence putting $m = m_1 + \mu$ we have by Taylor's theorem

$$w = tm_1 - \frac{t\mu^2}{2m_1} + \ldots$$

This series proceeds according to powers of μ/m_1, which lies between $-\alpha$ and β, and therefore vanishes in the limit, and therefore ultimately

$$w = tm_1 - \frac{t\mu^2}{2m_1}, \quad v = e^{t\left(m_1 - \frac{\mu^2}{2m_1}\right)}.$$

Now between the limits $(1-\alpha)m_1$ and $(1+\beta)m_1$ of m consecutive terms of the series Σv are ultimately equal, and we may replace Σv by $\int v\,dm$ or $\int v\,d\mu$; and the limits of μ are $-\alpha m_1$ and $+\beta m_1$, that is in the limit $-\infty$ and $+\infty$. Hence in the limit

$$\Sigma v = \int_{-\infty}^{\infty} v\,d\mu = \sqrt{\frac{2\pi m_1}{t}}\, e^{tm_1},$$

$$\Sigma u_m = t^{-\frac{1}{2}}(2\pi m_1)^{\frac{1-t}{2}} m_1^{-s+t} e^{tm_1}, \quad \ldots\ldots\ldots\ldots\ldots(D),$$

which may be expressed by (C) in terms of ρ.

The function of ρ to which the complete series Σu_m bears ultimately a ratio of equality when ρ is infinite is thus found, without the necessity of expressing the series in the first instance for general values of ρ by means of a definite integral.

The same method will evidently apply to the series whose general term is formed from (A) by integrations or differentiations with or without intervening multiplications by powers of x, since this process will merely introduce factors of the form $m + c$ into the numerator or denominator or both, and in passing to the limit for $\rho = \infty$ these factors may be put outside after writing m_1 for m.

(2) *On the lowering of the freezing point of tin caused by the addition of other metals.* By C. T. Heycock, M.A., King's College, and F. H. Neville, M.A., Sidney Sussex College.

When considering the researches of Raoult on the lowering of the freezing point of solutions, it occurred to us that the solution of one element in another might present the simplest case.

We have therefore determined the freezing point of pure tin and of tin containing known quantities of other metals. Tin was first chosen, because its melting point is fairly low and permitted the use of a mercurial thermometer, and also on account of its well-known power of dissolving many metals. As in the case of other liquids, the moment of incipient solidification is sharply indicated by surfusion. The experiments were made in a crucible consisting of a block of cast iron six inches high and five inches in diameter, having a cylindrical hole one inch in diameter and four inches long bored along its axis; this arrangement ensured slow cooling. Into this hole, which served as a cavity for containing

the metals, a known weight of tin (from 200 to 400 grams) was introduced together with some solid paraffin to prevent oxidation.

The block was then heated by means of a Bunsen; and a delicate thermometer, on which $\frac{1}{100}$ths of a degree centigrade by aid of a reading telescope could be estimated, was inserted; the molten tin was continually stirred by means of a plunging annular stirrer worked by a water motor. The stirring was continued during the whole time the tin cooled, and the highest temperature ever reached by the thermometer after the surfusion commenced was taken to be the exact temperature of solidification. The thermometers used in the earlier experiments were found to be quite unsuitable, as the high temperatures at which they were kept for many hours caused a perceptible rise in the zero point during the course of an experiment.

By using the fixed zero thermometers of Mr J. J. Hicks of Hatton Garden this difficulty was overcome, no shift in the zero being perceptible after many weeks of use.

With the single exception of antimony all the metals which we have at present been able to dissolve in tin cause a lowering in the freezing point, and further for the same metal this fall is proportional to the weight of the metal in solution and inversely as the atomic weight.

OBSERVED LOWERING OF FREEZING POINT OF TIN CAUSED BY ADDING ONE ATOMIC PROPORTION OF METAL TO ONE HUNDRED ATOMIC PROPORTIONS OF TIN (*i.e.* 11800).

	Approximate temperature of fusion.	Atomic weight.	Observed lowering.	Remarks.
Sodium	96	23	2·5	soluble with difficulty.
Aluminium	600	27	1·4	different samples of Aluminium.
Copper	1050	63	2·47	
Zinc	412	65	2·53	
Silver	954	108	2·67	very pure precipitated Ag.
Cadmium	320	112	2·16	
Gold	1180	197	2·65	very soluble.
Mercury	− 38·5	200	2·3	
Lead	335	208	2·22	
Bismuth	265	210	2·26	
Antimony	210	120	3·86 to 2·82	rise.

The preceding table gives a summary of the results hitherto obtained. Column four gives the lowering of the freezing point of tin caused by dissolving one atomic weight of each of the metals in 100 atomic weights of tin, and is calculated by proportion from the observed numbers.

The case of aluminium is of special interest, and perhaps points to the molecule of this metal being Al_2, that is, assuming that zinc, cadmium and mercury in dilute solution remain monatomic as in the gaseous state. We conclude from our experiments:

(a) That up to the limit of solubility of a metal in tin at the observed temperatures the lowering of the freezing point is directly proportional to the weight of metal added.

(β) That this lowering of the freezing point is inversely as the atomic weight of the metal added.

These laws hold also for the solution of medals in sodium, but the atomic fall instead of being about $2^{\circ}\cdot 4$ is about $4^{\circ}\cdot 5$ C.

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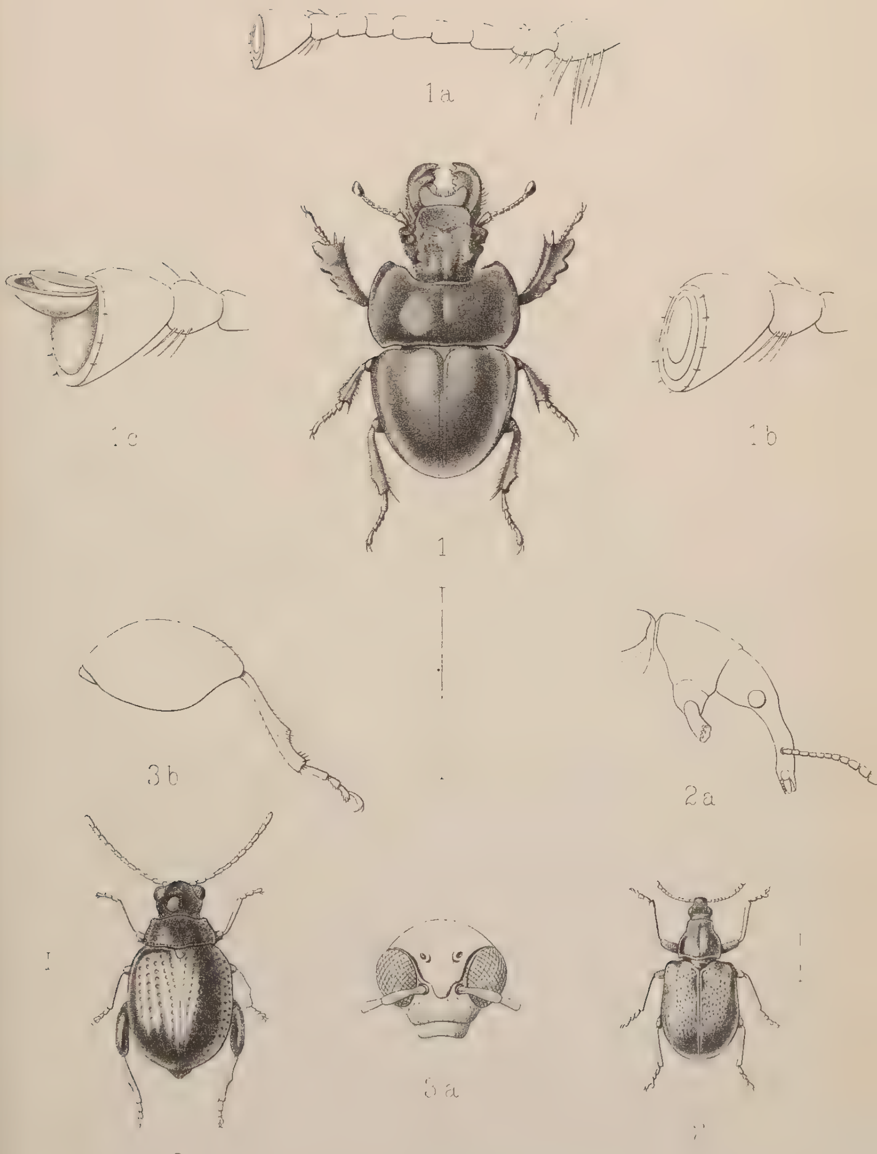

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